Robust Predictions in Dynamic Policy Games*

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Abstract

Dynamic policy games feature a wide range of equilibria. This paper provides a methodology for obtaining robust predictions. We begin by focusing on a model of sovereign debt although our methodology applies to other settings, such as models of monetary policy or capital taxation. The main result of the paper is a characterization of distributions over outcomes that are consistent with a subgame perfect equilibrium conditional on the observed history. We illustrate our main result by computing, conditional on an observed history, bounds on the maximum probability of a crisis, and bounds on means and variances over debt prices. In addition, we propose a general dynamic policy game and show how our main result can be extended to this general environment.

Keywords: multiple equilibria, robustness, moment inequalities, correlated equilibrium, policy games.

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1 Introduction

Following Kydland and Prescott (1977) and Calvo (1978), the literature on optimal government policy without commitment has formalized interactions between a large player (government) and a fringe of small players (households, lenders), dynamic policy games, by building on the tools developed in the work of Abreu (1988) and Abreu et al. (1990) in the literature of repeated games. This agenda has studied interesting applications for capital taxation (e.g., Chari and Kehoe, 1990, Phelan and Stacchetti, 2001, Farhi et al., 2012), monetary policy (e.g., Ireland, 1997, Chang, 1998a, Sleet, 2001) and sovereign debt (e.g., Calvo, 1988, Eaton and Gersovitz, 1981, Chari and Kehoe, 1993, Waki et al. (2018)) and helped us to understand the distortions introduced by lack of commitment and the extent to which governments can rely on reputation to achieve better outcomes.

One of the challenges in studying dynamic policy games is that these settings typically feature a wide range of equilibria with different predictions over observable outcomes. For example, there are "good" equilibria where the government may achieve, or come close to achieving, the optimum with commitment, while there are "bad" equilibria where this is far from the case, and the government may be playing the repeated static best response. When studying dynamic policy games, which of these should we expect to be played? Can we make general predictions given this pervasive equilibrium multiplicity? One approach is imposing refinements, such as various renegotiation-proof notions, that either select an equilibrium or significantly reduce the set of equilibria. Unfortunately, no general consensus has emerged on the appropriate refinements.

The goal of this paper is to overcome the challenge multiplicity raises by providing predictions in dynamic policy games that hold across all equilibria; following the terminology of Bergemann and Morris (2013), *robust predictions*. The approach we offer involves making predictions for future play that depend on past, observed play. The key idea is that even when little can be said about the *unconditional* path of play, quite a bit can be said once we *condition* on past observations. To the best of our knowledge, this simple idea has not been exploited as a way of deriving robust implications from the theory. Formally, we introduce and study a concept which we term "equilibrium consistent outcomes": outcomes of the game, after an observed history, that are consistent with some subgame perfect equilibria that on its path could have generated the observed history.

Although the notions we propose and results we derive are general and apply to a large class of dynamic policy games, for concreteness we first develop them for a specific application, using a model of sovereign debt along the lines of Eaton and Gersovitz (1981). In the model, a small open economy faces a stochastic stream of income. To smooth consumption, a benevolent government can borrow from international debt markets, but lacks commitment to repay. If it defaults on its debt, the only punishment is permanent exclusion from financial markets; it can never borrow again. There are two features of this model that make it appealing to our work. First, this model has been widely adopted and is a workhorse in international economics. Second, as we show in this paper, this policy game can feature wide equilibrium multiplicity. On one end of the spectrum, in the worst equilibrium, the government is in autarky, facing a price of zero for debt issuance, and consuming its income. Meanwhile, in the best equilibrium, the government smooths consumption, and there is no room for self-fulfilling crises.[1]

Our main result, Proposition 1, following the classic approach to study correlated equilibrium first proposed by Aumann (1987),[2] characterizes probability distributions over outcomes, what we term as *equilibrium consistent distributions*. Even though in the model any *equilibrium* price can be realized after a particular equilibrium history, we show that there are bounds on the probability distributions over these prices. For example, if the country just repaid a high amount of debt, or did so under harsh economic conditions, when output was low, the less likely are bad realizations of prices. The choice to repay under these conditions reveals an optimistic outlook for bond prices that narrows down the set of possible equilibria for the continuation game. This optimistic outlook is the expression of a *dynamic revealed preference* argument. What the government has left on the table as a consequence of its past decisions, reveals it's expectations over future play. In equilibrium, these expectations over the future must be correct, and hence imposes restrictions over future outcomes, which are the basis of the predictions we obtain in this paper.

Building on the characterization of equilibrium consistent distributions, we then turn to explore the predictions on observables that hold across all equilibria. In particular, we focus on debt prices. First, in Proposition 2, we obtain bounds on the maximum

[1]Given that our approach tries to overcome the challenges of multiplicity, we first ensure that there is multiplicity in the first place. In particular we show that in the standard Eaton and Gersovitz (1981) model, restrictions on debt, which are often adopted in the quantitative sovereign-debt literature (Chatterjee and Eyigungor 2012 and micro-founded in Amador, 2013), can imply the existence of multiple equilibria (see Auclert and Rognlie, 2016 for necessary and sufficient conditions for uniqueness). Our multiplicity relies on the existence of autarky as a subgame perfect equilibrium. This result may be of independent interest, since it implies that rollover crises are possible in this setting. The quantitative literature on sovereign debt following Eaton and Gersovitz (1981) features defaults on the equilibrium path, that are caused by shocks to fundamentals (see Stangebye (2019) for a recent exception). Another strand of the literature studies self-fulfilling debt crises following the models in Calvo (1988) and Cole and Kehoe (2000). Our results suggest that crises, defined as episodes where the interest rates are very high but not due to fundamentals, may be a robust feature in models of sovereign debt.

[2]More recently, this approached has been also adopted by the literature on information design. See Bergemann and Morris, 2018 for a review.

probability of low prices; for example, a rollover debt crises (i.e. a price realization of zero). Due to equilibrium multiplicity, rollover debt crises may occur on the equilibrium path for any realization of the fundamentals. However, the probability of a rollover crisis, after a certain history, may be constrained. We derive these constraints, showing that rollover crises are less likely if the borrower has recently made sacrifices to repay. Second, we use our characterization to obtain bounds on moments of distributions over outcomes. In particular, in Proposition 3, we characterize bounds over the expected value of debt prices given a history for any equilibrium. Third, in Proposition 4, as in Bergemann et al. (2015b), we characterize bounds on variances which hold across all equilibria. Finally, we propose a simple linear program that characterizes all non-centered moments over observables.

In the last section of the paper we show how our characterization of equilibrium consistent outcomes extends to a more general class of dynamic policy games. In particular, we provide a general model of credible government policies, which follows the seminal contribution of Stokey (1991). The key features that the general setup tries to capture are lack of commitment, a time inconsistency problem, infinite horizon that creates reputation concerns in the sense of trigger-strategy equilibria, and short run players that form expectations regarding the policies of the government. With some variation on the timing of the moves for the players, most dynamic policy games share these features. In particular, we show that the model of sovereign debt as in Eaton and Gersovitz (1981) and the New Keynesian model as in Woodford (2011) and Galí (2015) fit in this setup. In Proposition 6 we show how the main results of the paper extend into this general environment, and in subsection 4.2 we provide an additional application focusing on the New Keynesian model.

What is the importance of obtaining robust predictions? What we describe as robust predictions in this paper, which follows the terminology in Bergemann and Morris (2013), can also be described as the observable implications of equilibrium. In two influential papers, Jovanovic (1989) and Pakes et al. (2015), characterize for static games the observable implications of models with multiple equilibria. These implications, which are based on a *static revealed preference* argument, have been the basis of large literature in Industrial Organization and Econometrics that utilize them to estimate models with multiple equilibria (see Tamer, 2010 and De Paula, 2013 for recent reviews). To the best of our knowledge, ours is the first paper to obtain predictions over observables in a dynamic model with multiple equilibria without appealing to any equilibrium selection. We believe that our main results could be used as the basis of estimation techniques for dynamic models without imposing assumptions regarding the class of equilibria.

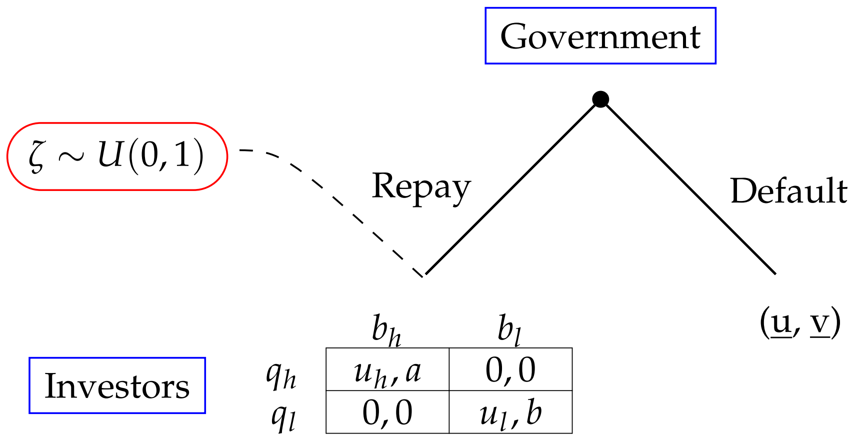

Figure 1: two period Example

Example and Main Results. We illustrate and provide a roadmap for our main results in a canonical and simple two period example.[3] Figure 1 depicts a two player game in which the government has the choice of defaulting (choosing $x =$ Default) and receiving a sure payoff of $\underline{u}$, or repaying debt (choosing $x =$Repay), and hence choosing to play a simultaneous move game G with the investors. If the government chooses $x =$ Repay, a public random variable $\zeta \sim$ Uniform $[0,1]$ (a *sunspot* variable) is observed by both parties before the subgame to be played between the government and the investors. The choices for the government (debt) and the investors (debt prices) in the coordination game are $(b_h, b_l) \in \mathbb{R}^2$ and $(q_h, q_l) \in \mathbb{R}^2$ respectively. The parametric assumptions are that $u_h, u_l, a, b > 0$ and $\underline{u} \in (u_l, u_h)$.[4]

Equilibrium. The coordination game has two equilibria in pure strategies: (b_l, q_l) and (b_h, q_h), which we will call the low equilibrium and high equilibrium. We can summarize any equilibrium outcome as a pair (x, Q), where $x \in \{\text{Repay}, \text{Default}\}$ is the decision of the government to play the coordination game or not, and $Q = (Q_l, Q_h)$ is a distribution over the low and high equilibrium; i.e. $Q_k = \Pr(\zeta : (b_k, q_k) \text{ is played})$ for $k \in \{l, h\}$, and $Q_l + Q_h = 1$.

Equilibrium Consistent Distributions. The main result of the paper, Proposition 1, characterizes distributions over observables after observing a equilibrium history of play. Lets delve into the intuition of this result. Suppose that we (as outsiders) observe that the

[3] A more formal discussion of this game is in the Online Appendix section A.

[4] The game that we study in this example is slightly different to the one that we study in Sections 2 and 4. The coordination game in the second step of the game depicted in Figure 1 tries to illustrate the inherent coordination over continuation play at the heart of repeated games, which is also the cause for the typical equilibrium multiplicity present in these games.

government has repaid debt. Both the high and low equilibrium are Nash equilibria of the static game. However, not all distributions over the high and low equilibrium could have been generated by a SPE. Thus, *the fact that some subgame perfect equilibria generated the history will place bounds over outcomes*. For example, there is no equilibrium that on its path generates $x = \text{Repay}$ and the government and the investors coordinate in the low equilibrium with probability one. The reason is that $x = \text{Repay}$ is not optimal for the government if they expect the low equilibrium with probability one. Following the same logic, we can dig deeper. In particular, the only equilibrium distributions consistent with $x = \text{Repay}$ are those that would have made it optimal for the government to plan $x = \text{Repay}$ in the first node. Those distributions $Q_l \in [0, 1]$ are characterized by the following condition:

$$u_h(1 - Q_l) + u_l Q_l \geq \underline{u}. \tag{1.1}$$

Equation (1.1) in fact defines the set of all possible distributions over outcomes that are equilibrium consistent with $x = \text{Repay}$. This sequential optimality of choices, is the main insight of Proposition 1, which is the main result of the paper.

Aided with (1.1) we can obtain bounds over moments of distributions. Obtaining these bounds is not computationally costly because they solve a *linear program*.

Bounding Moments: Probability of Crisis. What is the maximum probability of the low equilibrium after observing $x = \text{Repay}$? This boils to choose the maximum Q_l such that (1.1) holds. This value is equal to $\underline{Q}_l := (u_h - \underline{u}) / (u_h - u_l) \in (0, 1)$. This bound is intuitive. As the utility of the good equilibrium u_h increases, $\overline{Q}_l$ decreases. As the utility of default $\underline{u}$ increases, this probability decreases. We characterize this bound for the general model in Proposition 2.

Expectations. We can also bound other moments. As another example, we can obtain bounds on expectations of prices. Denote by $\mathbb{E}^Q(q)$ the expected value of the price q for any equilibrium consistent outcome $(x = \text{In}, Q)$. The upper bound, the maximum expectation is the one that corresponds to the largest probability of the high equilibrium. This probability distribution sets Q_l equal to zero, and has an associated expectation equal to q_h. The lowest expectation solves the following program

$$\underline{\mathbb{E}}^Q(q) = \min_{Q_l} Q_l q_l + (1 - Q_l) q_h$$

subject to (1.1). The solution of this program, and the fact that the largest expectation is q_h, defines a set of expected prices equal to $\left[\underline{\mathbb{E}}^Q(q), q_h\right]$, with $\underline{\mathbb{E}}^Q(q) = \left(1 - \underline{Q}_l\right) q_h + \underline{Q}_l q_l > q_l$. We use the same argument in Proposition 3, where we obtain precise bounds over expectations for the model of sovereign borrowing.

Variances. Once we know the set of all possible expected values of q across equilibria, we also can bound second moments. In particular, we can map distributions over prices q to pairs of expectations and variances $(\mathbb{E}(q), \mathbb{V}(q))$, where $\mathbb{V}(q)$ is the variance of q under some equilibrium distribution Q. In particular, given an expected price $\mu = \mathbb{E}(q) \in \left[\underline{q}, q_h\right]$, the maximum possible variance is $\left(1 - Q_l^{\mu}\right) q_h^2 + Q_l^{\mu} q_l^2 - \mu^2$, where $Q_l^{\mu} := (\mu - q_l) / (q_h - q_l)$. Again, this is the solution to a linear program, in which the objective is the variance, and the constraint is (1.1), and the fact that the mean of the distribution is equal to μ. In Proposition 4 we show that for the model of sovereign borrowing, the upper bound on variance always solves a linear programming problem as well, and actually can always be implemented by a distribution with only two prices in its support (even if q is a continuum).

Equilibrium Consistency vs. Forward Induction. It is important to distinguish equilibrium consistency from Forward Induction. The game depicted in Figure 1 is also useful for that. For concreteness, suppose that there is no sunspot (i.e. ζ is constant). In this game, the set of subgame perfect equilibria with forward induction has only one equilibrium in pure strategies $(x = \text{Repay}, (b_h, q_h))$. The subgame perfect equilibrium $(x = \text{Default}, (b_l, q_l))$ does not survive forward induction. But, because it is a subgame perfect equilibrium, it is equilibrium consistent. This example illustrates the main difference. Forward induction is a refinement on the set of equilibria; i.e., it shrinks the set of subgame perfect equilibria. Equilibrium consistency, on the other hand, does not shrink the set of equilibria, but rather introduces restrictions over observables.

Literature Review. Our paper relates to several strands of the literature. First, to the literature on credible government policies. The seminal papers on optimal policy without commitment are Kydland and Prescott (1977) and Calvo (1978). Applications range from capital taxation as in Phelan and Stacchetti (2001) and Farhi et al. (2012); monetary policy as in Ireland (1997), Chang (1998a), Sleet (2001) and Waki et al. (2018); and sovereign debt Atkeson (1991), Arellano (2008), Aguiar and Gopinath (2006), Cole and Kehoe (2000), and more recently Dovis (Forthcoming). We believe that our paper is closely related to Chari and Kehoe (1990), Stokey (1991) and Atkeson (1991). The first two papers adapt the techniques developed in Abreu (1988) to characterize completely the set of equilibria in dynamic policy games. Atkeson (1991) extends the techniques in Abreu et al. (1990), by allowing for a stochastic public state variable, in the context of sovereign lending finding interesting properties of the best equilibrium. Our paper studies a related, yet different question. Instead of characterizing equilibria at the ex-ante stage of the game in terms of sequences of observables, we provide a recursive characterization of the set of continuation equilibria given an equilibrium history of play. This characterization of continuation equilibria is precisely the basis for obtaining predictions that are robust across all equilibria. Our central assumption is that *an* equilibrium has generated the history of play, without appealing to any equilibrium refinement.

Second, to the literature on robust predictions. The papers that are more closely related to our work are Angeletos and Pavan (2013), Bergemann and Morris (2013), Bergemann et al. (2015b) and Chahrour and Ulbricht (2020) The first paper, Angeletos and Pavan (2013), obtains predictions that hold across every equilibrium in a global game with an endogenous information structure. The second paper, Bergemann and Morris (2013), obtains restrictions over moments of observable endogenous variables that hold across every possible information structure in a class of coordination games. In a related paper, Bergemann et al. (2015b) characterize bounds on output volatility, across all potential information structures, in a static model where agents face both idiosyncratic and common shocks to productivity. Our paper contributes to this literature by obtaining predictions that hold across all equilibria in a dynamic game. In particular, we obtain restrictions over the distribution of equilibrium debt prices, for any possible process of sunspots (potentially non-stationary), by exploiting the dynamic implications that sequential rationality has on the distribution of observables. These implications are the basis to obtain bounds on first and second order conditional moments, across all possible sunspot processes, or following the terminology in Bergemann and Morris (2018), across all possible *information structures*.

Chahrour and Ulbricht (2020) use this approach while extending their results to dynamic linear macroeconomic environments, where agents have access to arbitrary dynamic information structures about fundamental shocks and prices. The authors also obtain moment conditions on "wedges" that are akin to the results in Bergemann and Morris (2013) and ours as well, which allows them to obtain testable implications. In our paper, we instead focus on pure strategic uncertainty rather than payoff uncertainty. Also related is De Oliveira and Lamba (2019), where the authors obtain testable implications of bayesian rationality over a single agent choosing sequentially, but where agents may have access to an arbitrary dynamic information structures that could rationalize their behavior.[5]These bounds provide testable implications of the model, even in the presence

[5]The literature of information design in dynamic games, where agents may have access to private information about other players actions, was first formalized by Myerson (1986) and Forges (1986), extending the concept of correlated equilibrium of Aumann (1987) to extensive form games. As reviewed in Bergemann and Morris (2018), one can view the problem of information design from two alternative points of view. In the first one, the "literal interpretation", an information designer sends signals to other parties, to influence their behavior in order to achieve some objective. A large literature has grown after the contribution of Kamenica and Gentzkow (2011); see for example, on static environments, Gentzkow and Kamenica (2014),

of both equilibrium multiplicity and uncertainty of the information structure agents have when making their decisions.

Third, sections 2 and 3 of this paper study robust predictions in a dynamic policy game that builds on Eaton and Gersovitz (1981). This framework, and variations of it, have been extensively used to study sovereign borrowing. The literature has followed two main directions. One direction, the quantitative literature on sovereign debt, following the initial contributions of Aguiar and Gopinath (2006) and Arellano (2008), studies sovereign spreads, debt capacity and welfare from a positive and normative point of view. The focus is usually on Markov equilibria on payoff relevant state variables and hence defaults can only be consequence of bad fundamentals. Our paper shares with this strand of the literature the focus on a model along the lines of Eaton and Gersovitz (1981) but rather than characterizing a particular equilibrium, we study predictions across all equilibria. In addition, we provide a full characterization of the set of equilibria and conditions for equilibrium multiplicity that are novel in the literature. The second direction focuses on equilibrium multiplicity, and in particular, on self fulfilling debt crises. The seminal contributions are Calvo (1988) and Cole and Kehoe (2000). Our paper studies multiplicity in an alternative setup, the one of Eaton and Gersovitz (1981);[6] the crucial difference between the setting in Cole and Kehoe (2000) and the one in Eaton and Gersovitz (1981) is that in the latter the government issues debt (with commitment) and then the price is realized, changing the source of equilibrium multiplicity. Our contribution to this strand of the literature is that by providing sufficient conditions for equilibrium multiplicity in a model as in Eaton and Gersovitz (1981) we show that once we introduce coordination devices, under the right parametric assumptions, coordination failures are a robust feature in models of sovereign lending.

Bergemann et al. (2015a), Gentzkow and Kamenica (2016), Duffie et al. (2017), Kolotilin et al. (2017), Inostroza and Pavan (2020); and for dynamic environments, see for example Doval and Ely (2020) and Ely et al. (2015). In the second one, the "metaphorical interpretation", the designer is an abstraction that chooses among different information structures to achieve some objective. For example, in Bergemann et al. (2015b), the "objective" of the designer is to maximize output volatility. The literature on robust predictions falls in this category; see for example Benoît and Dubra (2011), Burks et al. (2013), Bergemann and Morris (2013). Our paper, of course, belongs to the second interpretation. Finally, Sugaya and Wolitzky (2020), links the two points of view, under certain conditions (information designer or "mediator" can tremble).

[6] A recent exception that studies multiplicity in a model as in Eaton and Gersovitz (1981) is Stangebye (2019). Auclert and Rognlie (2016) find necessary and sufficient conditions for uniqueness in a model as in Eaton and Gersovitz (1981). Recent contributions to the strand of the literature that studies defaults due to fundamentals, among many others, are Chatterjee and Eyigungor (2015), Hatchondo et al. (2016), Pouzo and Presno (2016), Arellano and Bai (2014), Arellano and Bai (2017), Ottonello and Perez (2019), Aguiar et al. (2020), Bianchi et al. (2018), Passadore and Xu (2018) and Sánchez et al. (2018). Recent contributions to the strand that studies equilibrium multiplicity, following Calvo (1988) and Cole and Kehoe (2000), are Lorenzoni and Werning (2018), Bocola and Dovis (2019), Aguiar et al. (2020), Corsetti and Dedola (2016), Roch and Uhlig (2018), and Ayres et al. (2018). See Aguiar and Amador (2013) for a comprehensive review.

Finally, our paper relates to the literature that studies the observable implications of models with multiple equilibria. The two more closely related papers are Jovanovic (1989) and Pakes et al. (2015). The first paper, Jovanovic (1989), provides a framework to discuss conditions under which a model with multiple equilibria is point or set identified. The main ideas are clearly illustrated in a two person entry game, one of the canonical examples of estimation of games with multiple equilibria.[7] The second paper, Pakes et al. (2015), discuses conditions under which inequality constraints can be used as a basis for estimation and inference.[8] Both papers are based on a revealed preference argument that places bounds over observables given an optimizing behavior of an agent. Our paper, is based on a dynamic version of this revealed preference argument: what the government just left on the table, reveals an outlook for the future, and this outlook for the future places bounds over observables. The importance of obtaining dynamic observable implications is that extends the applicability of the previous results, which focus on a static setting.

Outline. The paper is structured as follows. Section 2 introduces the model. Section 3 discusses the characterization of equilibrium consistent outcomes. Section 4 spells out the general model and states the main results of the paper in this setup. Section 5 concludes.

2 A Dynamic Policy Game

Our model of sovereign debt follows Eaton and Gersovitz (1981). Time is discrete and denoted by $t \in \{0, 1, 2, ...\}$. A small open economy receives a stochastic stream of income denoted by y_t. Income follows a Markov process with c.d.f. denoted by $F(y_{t+1} \mid y_t)$. The c.d.f. $F(y_{t+1} \mid y_t)$ is non-atomic (y_t is an absolutely continuous random variable). There is a public randomization device, $\zeta_t \sim U[0, 1]$, i.id. over time. The government is benevolent and seeks to maximize the utility of the households. It does so by selling bonds, denoted by b_t, in the international bond market. The household evaluates consumption

[7] Entry games have been studied extensively in the IO literature (see for example Bresnahan and Reiss, 1990, Berry, 1992, Bajari et al., 2007, Ciliberto and Tamer, 2009), or are examples of a large literature on estimation of static and dynamic games of complete (see for example Aguirregabiria and Mira, 2007 and Bajari et al., 2010) and incomplete (see for example De Paula and Tang, 2012) information.

[8] Moment conditions that yield inequality constraints, as observable implications of equilibria, have spurred a literature in econometrics that studies inference and consistency of structural estimates that are based on moment inequalities (see for example, Chernozhukov et al. 2007, Beresteanu et al. 2011, Bugni, 2010, Romano and Shaikh, 2010), or that estimates structural parameters in games with multiple equilibria (see for example Ciliberto and Tamer, 2009, among others). Identification of structural parameters is also a part of a much larger literature on partial identification in econometrics (see for example Tamer, 2010 for a recent review).

streams according to

$$\mathbb{E}\left[\sum_{t=0}^{\infty}\beta^t u(c_t)\right]$$

where $\beta < 1$ and u is increasing and strictly concave. The sovereign government issues short term debt at a price q_t. The budget constraint is

$$c_t = y_t - b_t + q_t b_{t+1}.$$

There is limited enforcement of debt. Therefore, the government will repay only if it is more convenient to do so. We assume that after a default the government remains in autarky forever after but there are not direct output costs of default. Furthermore, we also assume that the government cannot save:

$$b_{t+1} \geq 0.$$

The assumption of no savings, which implicitly captures political economy constraints that make it difficult for governments to save as modeled by Amador (2013), in addition to the assumption of no direct costs of default, is sufficient to guarantee that autarky is an equilibrium. The idea is that, if the government cannot save, and there are no output costs of default, if the government expects a zero bond price for its debt now and in every future period, then it will default its debt. To guarantee multiplicity we need to introduce conditions to guarantee that there is at least another equilibrium that has a positive debt capacity. In our paper, this equilibrium with a positive price of debt is the Markov equilibrium that is usually studied in the literature of sovereign debt.[9]

Lenders. There is a competitive fringe of risk neutral investors that discount the future at a rate of $r > 0$. This discount rate, and the possibility of default, imply that the price of the bond is given by

$$q_t = \frac{1-\delta_t}{1+r} \tag{2.1}$$

where δ_t if the default probability on bonds b_{t+1} issued at date t.[10]

[9]As we discuss in the Online Appendix B, and shown in Auclert and Rognlie 2016, no savings, $b_{t+1} \geq 0$, is a necessary condition for equilibrium multiplicity. One of the contributions in our paper is to show that, no savings, plus a set of parametric conditions are sufficient for equilibrium multiplicity. Another paper studying multiplicity in the Eaton and Gersovitz (1981) setup is Stangebye (2019). The setup in the latter differs from ours since there is long term debt and there are direct costs of default. In addition, the paper focuses on numerical results.

[10]This can be microfounded by a fringe of strategic agents who decide to lend b_{t+1} dollars to maximize expected profits $V = -q_t b_{t+1} + (1-\delta_t)\frac{1}{1+r}b_{t+1}$. If agents compete perfectly in the lending market, equa-

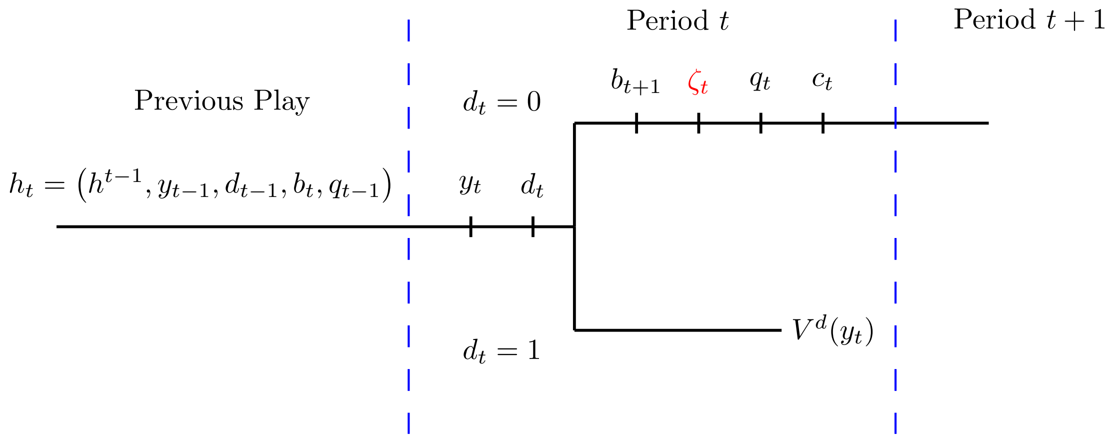

Figure 2: The figure summarizes the timing and the construction of histories in the case in which there is a sunspot. Now, we introduce a sunspot ζ_t after the government has issued debt b_{t+1} and before the price q_t has been realized.

Timing. The sequence of events within a period is as follows. In period t, the government enters with b_t bonds that it needs to repay. Then income y_t is realized. The government then has the option to default $d_t \in \{0, 1\}$. If the government does not default, the government runs an auction of face value b_{t+1}. A sunspot ζ_t is realized. Then, the price of the bond q_t is realized. Finally, consumption takes place, and is given by $c_t = y_t - b_t + q_t b_{t+1}$. If the government decides to default, then consumption is equal to income, $c_t = y_t$. The same is true if the government has ever defaulted in the past.

Histories, Strategies, and Outcomes. A *history* is a vector $h^t = (h_0, h_1, ..., h_{t-1})$, where $h_t = (y_t, d_t, b_{t+1}, \zeta_t, q_t)$ is the outcome of observable variables of the stage game at time t. A partial history is an initial history h^t concatenated with a history of the stage game at period t. For example, $h^t_g = \left(h^t, y_t\right)$ is a history after which the government must choose policies (d_t, b_{t+1}). The set of all partial histories is denoted by $\mathcal{H}$. We label as $\mathcal{H}_g \subset \mathcal{H}$ the partial histories where the government has to choose policies. Likewise, $\mathcal{H}_m \subset \mathcal{H}$ is the set of partial histories where the market plays; for example, $h^t_m = \left(h^t, y_t, d_t, b_{t+1}\right)$. A policy maker's strategy is a function $\sigma_g\left(h^t, y_t\right) = \left(d^{\sigma_g}_t, b^{\sigma_g}_{t+1}\right)$ for all histories $\left(h^t, y_t\right) \in \mathcal{H}_g$. A strategy for the market is a pricing function $q_m\left(h^t, y_t, d_t, b_{t+1}\right)$ for all histories $h^t_m \in \mathcal{H}_m$. Denote by Σ_g and Σ_m the set of strategies for the government and the market. For a strategy profile $\sigma = \left(\sigma_g, q_m\right)$ we write $V\left(\sigma \mid h\right)$ for the continuation expected utility, after

tion 2.1 is derived as a non-arbitrage equilibrium condition. See for example Arellano (2008).

history h, of the representative consumer if agents play according to profile σ. For any strategy profile $\sigma \in \Sigma := \Sigma_g \times \Sigma_m$, we define the continuation at $h_g^t \in \mathcal{H}_g$

$$V(\sigma \mid h_g^t) = \mathbb{E}_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[(1-d_s) u(y_s - b_s + q_s b_{s+1}) + d_s u(y_s) \right] \right\}$$

where (y_s, d_s, b_{s+1}, q_s) are generated by the strategy profile σ.[11]

Equilibrium. A strategy profile $\sigma = (\sigma_g, q_m)$ constitutes a *subgame perfect equilibrium (SPE)* if and only if, for all partial histories $h_g^t \in \mathcal{H}_g$

$$V(\sigma \mid h_g^t) \geq V(\sigma_g', q_m \mid h_g^t) \text{ for all } \sigma_g' \in \Sigma_g, \tag{2.2}$$

and for all histories (h_m^t, ζ_t)

$$q_m\left(h_m^t, \zeta_t\right) = \frac{1}{1+r} \mathbb{E}_t \left(1 - d^{\sigma_g}(h^{t+1}, y_{t+1})\right). \tag{2.3}$$

That is, the strategy of the government is optimal given the pricing strategy of the lenders $q_m(\cdot)$; likewise, $q_m(\cdot)$ is consistent with the default policy generated by σ_g. The set of all subgame perfect equilibria is denoted as $\Sigma^* \subset \Sigma$. Given any history $h \in \mathcal{H}$, denote $\Sigma^*(h)$ as the set of all equilibrium strategies begining at history h.

Equilibrium Prices, Continuation Values. For any history h_m^t we define the highest and lowest prices *equilibrium* prices as:

$$\overline{q}^E(h_m^t) := \max_{q_m \in \Sigma^*(h_m^t)} q_m\left(h_m^t\right) \tag{2.4}$$

$$\underline{q}^E(h_m^t) := \min_{q_m \in \Sigma^*(h_m^t)} q_m\left(h_m^t\right). \tag{2.5}$$

In the Online Appendix, Section B, we describe necessary and sufficient conditions for equilibrium multiplicity.[12] In addition, we show that the worst SPE price is zero (i.e.,

[11]Note that expectation is taken with respect to the probability distribution of the stochastic processes of output and the sunspot, given the strategy for both the market and the government. We sometimes use $b_s = b_s^{\sigma_g}$ and $d_s = d_s^{\sigma_g}$ for clarity.

[12]There are two points worth noting. First, our analysis may be of independent interest, because we describe conditions under which there are multiple Markov equilibria in a sovereign debt model that follows Eaton and Gersovitz (1981), a framework that has been widely adopted in the literature. The importance of this result is that it opens up the possibility of confidence crises in a class of models that are usually utilized to study crises that are due to bad fundamentals. Thus, confidence crises are not necessarily a special fea-

$\underline{q}^E(h_m^t) = 0$) and the associated equilibrium payoff is given by the utility level of autarky. The lowest price $\underline{q}^E(h_m^t)$ is attained by using a fixed strategy for all histories (default after any history). The level of utility of autarky is given by:

$$V^d(y_t) := u(y_t) + \beta \mathbb{E}_{y_{t+1}|y_t} V^d(y_{t+1}). \tag{2.6}$$

Alternatively, the highest price $\overline{q}^E(h_m^t)$ is associated with a, different, fixed strategy for all histories, is Markov in (b_t, y_t) conditional on no default so far, and delivers the highest equilibrium level of utility for the government. We denote the best equilibrium price as $\overline{q}^E(h_m^t) = \overline{q}(y_t, b_{t+1})$. In the Online Appendix, Section B, we show that this price is associated with the best equilibrium in terms of welfare. The continuation utility (conditional on not defaulting) of the choice b_{t+1} given bonds and output (b_t, y_t) in the best equilibrium is given by

$$\overline{V}^{nd}(b_t, y_t, b_{t+1}) := u(y_t - b_t + \overline{q}(y_t, b_{t+1}) b_{t+1}) + \beta \overline{\mathbb{V}}(y_t, b_{t+1}), \tag{2.7}$$

where $\overline{\mathbb{V}}(y_t, b_{t+1})$ is defined as

$$\overline{\mathbb{V}}(y_t, b_{t+1}) := \mathbb{E}_{y_{t+1}|y_t}\left[\max\left\{\overline{V}^{nd}(b_{t+1}, y_{t+1}), V^d(y_{t+1})\right\}\right], \tag{2.8}$$

and $\overline{V}^{nd}(b_t, y_t) := \max_{b' \geq 0} \overline{V}^{nd}(b_t, y_t, b_{t+1})$. Aided with the previous definitions, the best equilibrium price is defined as $\overline{q}(y_t, b_{t+1}) := \frac{\mathbb{E}_{y_{t+1}|y_t}[1 - d(y_{t+1}, b_{t+1})]}{1+r}$ where $d(y_{t+1}, b_{t+1})$ is equal to zero if and only if $\overline{V}^{nd}(b_{t+1}, y_{t+1})$ is greater than of equal to $V^d(y_{t+1})$.

Ex-Post Best Continuation Value. The maximum continuation value function $\overline{v}(y_t, b_{t+1}, q_t)$ given bonds b_{t+1}, issued at a price q_t, when income is y_t, is defined as:

$$\overline{v}(y_t, b_{t+1}, q_t) := \max_{\sigma \in \boldsymbol{\Sigma}^*(y_t, b_{t+1})} V(\sigma \mid y_t, b_{t+1}, q_t).$$

In the Online Appendix Section C we show that this function can be computed as:

$$\overline{v}(y_t, b_{t+1}, q_t) = \max_{d(\cdot) \in \{0,1\}^Y} \mathbb{E}_{y_{t+1}|y_t}\left[d(y_{t+1}) V^d(y_{t+1}) + (1 - d(y_{t+1})) \overline{V}^{nd}(b_{t+1}, y_{t+1})\right] \tag{2.9}$$

ture of the timing in Calvo (1988) and Cole and Kehoe (2000) but rather a robust feature of most models of sovereign debt. Second, given our assumptions of no savings and no direct costs of default, characterizing the equilibrium set is relatively straightforward.

subject to

$$q_t = \frac{\mathbb{E}_{y_{t+1}|y_t}\left(1 - d\left(y_{t+1}\right)\right)}{1+r}.$$

We also show that $\overline{v}\left(y_t, b_{t+1}, q_t\right)$ is non-increasing in b_{t+1}, and non-decreasing and concave in q_t.[13] The fact that the function is non-decreasing in q_t is intuitive: better prices are associated with better continuation equilibria, as well as higher contemporaneous consumption (since $b_{t+1} \geq 0$). Concavity follows from the fact that $\overline{v}\left(y_t, b_{t+1}, q_t\right)$ solves a linear maximization problem. We use both properties to obtain sharper characterizations of the set of equilibrium consistent distributions and to obtain testable predictions.

3 Equilibrium Consistency

We now introduce the concept of equilibrium consistency. Given a SPE profile $\sigma = \left(\sigma_g, q_m\right)$, we define its *equilibrium path* $x\left(\sigma\right)$ as a sequence of measurable functions $x\left(\sigma\right) = \left(d_t^{\sigma_g}\left(\zeta^{t-1}, y^t\right), b_{t+1}^{\sigma_g}\left(\zeta^{t-1}, y^t\right), q_t^{q_m}\left(y^t, \zeta^t\right)\right)_{t \in \mathbb{N}}$ that are generated by following the profile σ.

Definition 1. A history $h \in \mathcal{H}$ is *equilibrium consistent* if and only if it is on the support of some equilibrium path $x\left(\sigma\right)$, for some SPE profile σ.

Definition 2. An observable outcome $x_t \subseteq \left(y_t, d_t, b_{t+1}, q_t\right)$ is equilibrium consistent with history h^t if there exists an equilibrium path that could generate it.

3.1 A Characterization

We are interested in characterizing equilibrium consistent prices and their distributions. Formally, a distribution of debt prices $Q_t \in \Delta\left(\mathbb{R}_+\right)$ is *equilibrium consistent* with history h_m^t if and only if for any Borel measurable set of prices $A \subseteq \mathbb{R}_+$ we have that $Q_t\left(A\right) = \Pr\left(\zeta_t : q_m\left(h_m^t\right) \in A\right)$ for some $q_m \in \Sigma_m^*\left(y_t, b_{t+1}\right)$. Given an equilibrium history $h_m^t = \left(h^t, y_t, d_t, b_{t+1}\right)$, and given an equilibrium strategy $\sigma = \left(\sigma^g, q^m\right)$, denote the set of *equilibrium consistent price distributions* as $\mathbb{ECD}\left(h_m^t\right)$. The following proposition characterizes this set.

[13] We reelegate the details to Appendix C. We will use interchangeably the notation $\overline{v}\left(y_{t-1}, b_t, q_{t-1}\right)$ or $\overline{v}\left(y_t, b_{t+1}, q_t\right)$, depending on what is more convenient. Note that the set of equilibrium strategies given history h^t, which we denote by $\Sigma^*(h^t)$, only depends on the initial bonds, b_t, and the seed value of income, y_{t-1}. Thus, $\Sigma^*(h^t) = \boldsymbol{\Sigma}^*\left(y_{t-1}, b_t\right)$. In the case of i.i.d. income, we can also drop the dependence on income, and therefore $\Sigma^*(h^t) = \boldsymbol{\Sigma}^*\left(b_t\right)$.

Proposition 1. *Suppose that h_m^t, with no default so far, is equilibrium consistent. Then, the triple $(d_t = 0, b_{t+1}, Q_t)$, where $Q_t \in \Delta(\mathbb{R}_+)$, is an equilibrium consistent outcome if and only if:*

(a) Debt prices are SPE prices; i.e.

$$Q_t \in \Delta\left(\left[0, \overline{q}\left(y_t, b_{t+1}\right)\right]\right) \tag{3.1}$$

(b) IC of the government:

$$\int \left[u\left(y_t - b_t + q_t b_{t+1}\right) + \beta \overline{v}\left(y_t, b_{t+1}, q_t\right)\right] dQ_t\left(q_t\right) \geq V^d\left(y_t\right). \tag{3.2}$$

Proof. See Appendix A. □

The main contribution of this paper is condition (3.2). Note that condition (3.1) characterizes prices that are SPE outcomes. Debt prices are between zero, and the best equilibrium price $\overline{q}(y_t, b_{t+1})$. The idea is that, if we do not assume that the history h_m^t is generated by some SPE, then there are no restrictions over debt prices other than being equilibrium prices.

The idea of the proof of Proposition 1 is as follows. For necessity, fix an equilibrium consistent distribution Q after history h_m^t. If we assume that h_m^t is on the equilibrium path of some SPE, then the government strategies, d_t and b_{t+1}, were optimal before the realization of the sunspot ζ_t. This implies that the government ex-ante preferred to pay the debt (i.e. $d_t = 0$) and issue bonds (b_{t+1}) rather than defaulting on the debt. If, after these decisions the realized price is q_t, the payoff for the government would be *at mos*t $u(y_t - b_t + q_t b_{t+1})$ plus the best ex-post continuation value $\overline{v}(y_t, b_{t+1}, q_t)$. However, the government is uncertain over which price will be realized for the debt issued. So, the government forms an expectation with respect to the "candidate" equilibrium consistent distribution Q. This expectation, and its associated expected utility, has to be at least as good as defaulting; if not, the government would have defaulted instead of repaying. The left hand side of condition (3.2) is an upper bound on the utility of not defaulting at history h_m^t. Thus, (3.2) is necessary. In other words, if it were to be violated, then we could not construct promises that rationalize the past history h_m^t.[14]

The idea of sufficiency, which is the reason why we eliminate b_{t-1} and all the previous policies, stems from the fact that both the output and the sunspot are non-atomic.[15]

[14]One might wonder why we cannot rely on on the best continuation payoff $\overline{\mathbb{V}}(y_t, b_{t+1})$. This is because this payoff is associated with the best equilibrium price, and this price needs not to be realized. The best possible payoff, after the price q is realized, is precisely $\overline{v}(y_t, b_{t+1}, q)$.

[15]Even if output where discrete, sunspots make shocks non-atomic, having the same effect as if we had absolutely continuous output shocks.

The particular history that followed h_m^{t-1} when b_{t-1} was chosen, the one with the particular realization of ζ_{t-1}, had zero probability of occurring, because the sunspot has a continuous distribution. Thus, it could always have been the case that the payoffs that rationalized b_{t-1} and the previous policies were to be realized in a state that never materialized. Therefore, $\mathbb{ECD}\left(b_t, y_t, b_{t+1}\right) = \mathbb{ECD}\left(h_m^t\right)$.

Proposition 1 can be specialized to obtain robust predictions over certain subset of subgame perfect equilibrium. For example, the result can be adapted for equilibria with limited punishment off equilibrium. Namely, the same results would hold if we replace $V^d\left(y\right)$ by a higher off-equilibrium payoff $V > V^d\left(y\right)$ in 2.9; for example, if agents are punished with default for a fixed or random number of periods.

Aided with Proposition 1, we now further characterize moments over distributions of debt prices.

3.2 Bounding Equilibrium Prices

Before bounding moments over distributions of prices we characterize the best continuation prices for the case without sunspots; i.e. ζ_t is constant. We term them equilibrium consistent prices. First, for each (b_t, y_t, b_{t+1}), we define the lowest equilibrium consistent price, $\underline{q}(b_t, y_t, b_{t+1})$, as:

$$u\left(y_t - b_t + \underline{q}b_{t+1}\right) + \beta\overline{v}\left(y_t, b_{t+1}, \underline{q}\right) = V^d\left(y_t\right). \tag{3.3}$$

Note that $\underline{q}$ is a function that maps $\underline{q}(b_t, y_t, b_{t+1}) : B \times Y \times B \to \left[0, \frac{1}{1+r}\right]$. Note also that $\underline{q}$ is unique, due to the monotonicity of $u(\cdot)$ and $\overline{v}\left(y_t, b_{t+1}, \cdot\right)$. The lowest equilibrium consistent price, $\underline{q}$, is the lowest price for debt issued b_{t+1}, given a debt payment b_t under and income realization y_t, for which the government does not default. Second, we can also define the highest equilibrium consistent price. It is given by $\overline{q}(y_t, b_{t+1})$, and is equal to the best equilibrium price defined in (2.4). The idea is that for any equilibrium history, the best equilibrium is a possible continuation equilibrium. In fact, the best equilibrium is not a possible continuation, then the previous history cannot be an equilibrium history. Now, we show some properties about these prices.

Corollary 1. *Let $\underline{q}\left(b_t, y_t, b_{t+1}\right)$ be the lowest equilibrium consistent price after history h_m^t. The following holds: (a) $\underline{q}\left(b_t, y_t, b_{t+1}\right)$* is decreasing in b_{t+1}; *(b) $\underline{q}\left(b_t, y_t, b_{t+1}\right)$ is increasing in b_t; and (c) For every equilibrium $\left(b_t, y_t, b_{t+1}\right)$, $-b_t + \underline{q}\left(b_t, y_t, b_{t+1}\right) b_{t+1} \leq 0$; if income is i.i.d., then $\underline{q}$ is decreasing in y_t, and so is the set of equilibrium consistent prices $[\underline{q}(b_t, y_t, b_{t+1}), \overline{q}(y_t, b_{t+1})]$.*

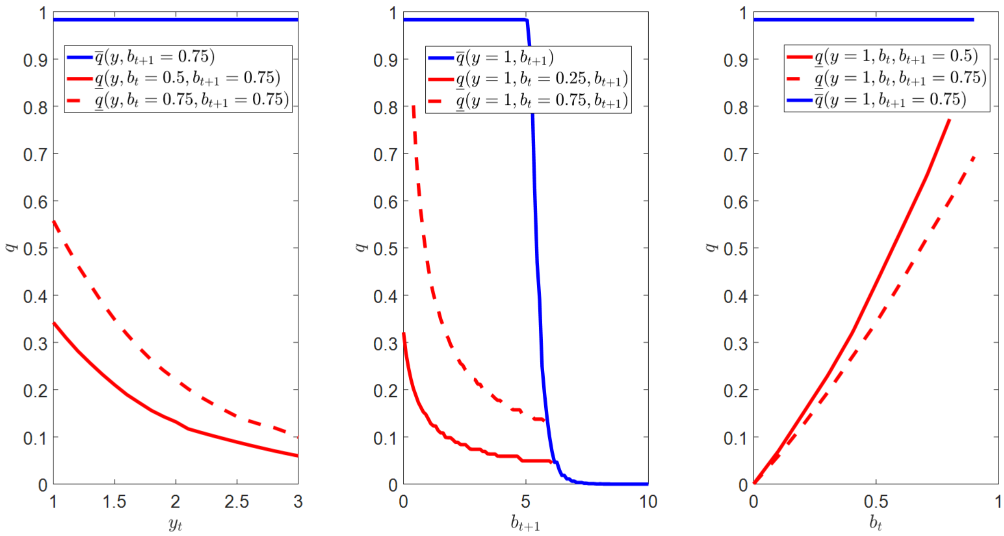

Figure 3: This figure plots equilibrium consistent prices $\overline{q}$ and $\underline{q}$. We describe the comparative statistics after history h_m^t. Thus, the relevant state variables are (b_t, y_t, b_{t-1}).

The intuition follows. First, note that the lowest equilibrium consistent price is decreasing in the amount of debt issued b_{t+1}. Higher amounts of debt issued imply a higher spot utility and a lower best continuation. However, the sum of them increases weakly. Therefore, $\underline{q}$ needs to decrease weakly. In other words, the past choices of the government could be rationalized with a lower price for the debt b_{t+1}. The opposite intuition holds for b_t; if the country just repaid a large amount of debt (i.e., made an effort to repay the debt), then the past choices are rationalized by using higher prices. Second, note that a positive capital inflow obtained at the lowest equilibrium consistent price would imply that $u(y_t) - u\left(y_t - b_t + \underline{q}(b_t, y_t, b_{t+1})\, b_{t+1}\right)$ is negative. Intuitively, the country is not making any effort to repay the debt. Therefore, it need not be the case that the country expects high prices for debt in the next period. Finally, because there are no capital inflows at the lowest equilibrium consistent price, repaying debt at this price will become more costly for a lower realization of income y_t; this due to the concavity of the utility function. Mathematically, because of concavity, $u(y_t) - u\left(y_t - b_t + \underline{q}(b_t, y_t, b_{t+1})\, b_{t+1}\right)$ is increasing as income decreases, and therefore, the promise-keeping constraint tightens as income decreases.

A Quantitative Illustration. We now numerically solve for the equilibrium consistent prices. The process for log output is given by $\log y_t = \mu + \rho_y \log y_{t-1} + \sigma_y \epsilon_t$ where $\mu = 0.75$, $\sigma_y = 0.3025$, ϵ_t is i.i.d. and $\epsilon_t \sim N(0,1)$, and $\rho_y = 0.0945$. The risk free interest rate is set to $r = 0.017$. The utility function is $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, the coefficient of relative risk aversion is $\gamma = 2$, and the discount factor $\beta = 0.953$.[16] Figure 3 depicts the numerical results. As we discussed before, the best equilibrium, $\overline{q}$, coincides with the equilibrium usually studied in the quantitative literature of sovereign debt. We plot the best equilibrium consistent price in blue and the lowest in red. As clearly shown in the Figure, for low levels of debt the best equilibrium is risk-free (default). As we increase the level of debt, the price drops, and prices drop sharply, as it is in most models with short-term debt (prices are volatile). The lowest equilibrium consistent price $\underline{q}(b_{t-1}, y_{t-1}, b_t)$ is computed using (2.9) and (3.3). Note that the comparative statics that we specified in the Corollary 1 clearly emerge in Figure 3. First, in the left panel, when the government repays debt $b_t = 0.5$ and issues $b_{t+1} = 0.75$, the lowest equilibrium consistent price decreases with the realization of income. This result occurs because higher levels of y_t imply that the government repaid under more favorable conditions. In addition, as one would expect, when the amount of debt repaid climbs to $b_t = 0.75$ and the amount of debt issued is still $b_{t+1} = 0.75$, the red dotted line dominates the red line. The lowest equilibrium consistent price is now higher. Finally, note that the best equilibrium price is constant through the realizations of income, because for those levels of debt, $b_{t+1} = 0.75$, default is not a concern. Also, note that in the right panel we observe that with debt repayment, b_t, we obtain the opposite: when the government repays a larger amount of debt, then the lowest equilibrium consistent price increases. This is the case for both $(y_t = 1, b_{t+1} = 0.50)$ and $(y_t = 1, b_{t+1} = 0.75)$.[17] The dotted line corresponds to a higher debt issuance, and as we just discussed, given a larger capital inflow, the prices are expected to be lower.

3.3 Bounding Price Distributions

We now delve into the implications of Proposition 1 on distributions over prices q_t. The first set of implications are over the probability of low prices. In particular, we characterize the maximum probability that a crisis will occur. Second, we provide bounds across all equilibria for the expectation of prices. Third, we also provide bounds across all equilibria for the variance of distributions over prices.[18] Finally, to close this subsection,

[16]We set the same parameters values for all the numerical exercises in this section, Section 3 and in the Online Appendix Section B.

[17]This result may be contrasted with the result in Cole and Kehoe (2000). In their setting the potential for rollover crises induces the government to lower debt below a threshold that rules rollover crises out. Thus, the government's efforts have no effect in the short run, but payoff in the long run. In our model, an outside observer will witness that rollover crises are less likely immediately after an effort has been made to repay the debt.

[18]All of these bounds are independent of the nature of the sunspots (i.e. the distribution of sunspots, its dimensionality, and so on), in the same way as the set of correlated equilibria does not depend on the actual

we study the comparative statistics for the set of equilibrium consistent distributions, $\mathbb{ECD}(b_t, y_t, b_{t+1})$.

Probability of Crises and the Infimum Distribution. We would like to infer the maximum probability (across equilibria) that the government *could* assign to a price $\hat{q}$ in any equilibrium after an equilibrium history h_m^t. Formally, we define the function $\underline{Q}(\hat{q})$ as:

$$\underline{Q}(\hat{q}; b_t, y_t, b_{t+1}) := \max_{Q \in \mathbb{ECD}(b_t, y_t, b_{t+1})} \Pr_Q(q \leq \hat{q}) \tag{3.4}$$

where $\Pr_Q(q \leq \hat{q}) := \int_0^{\hat{q}} dQ(q)$. The following proposition characterizes $\underline{Q}(\cdot)$.

Proposition 2. *Consider an equilibrium consistent history $h_m^t = (h^t, y_t, d_t = 0, b_{t+1})$. (a) For any $\hat{q} \geq \underline{q}(b_t, y_t, b_{t+1})$, $\underline{Q}(\hat{q}; b_t, y_t, b_{t+1}) = 1$. (b) For any $\hat{q} < \underline{q}(b_t, y_t, b_{t+1})$ it holds that:*

$$\underline{Q}(\hat{q}; b_t, y_t, b_{t+1}) = \frac{\overline{V}^{nd}(b_t, y_t, b_{t+1}) - V^d(y_t)}{V^d(y_t) - [u(y_t - b_t + \hat{q}b_{t+1}) + \beta\overline{v}(y_t, b_{t+1}, \hat{q})] + \overline{V}^{nd}(b_t, y_t, b_{t+1}) - V^d(y_t)}. \tag{3.5}$$

Proof. See Appendix A. □

The idea of the proof is as follows. Lets us start with the case $\hat{q} \geq \underline{q}(b_t, y_t, b_{t+1})$. The reason why $\underline{Q}(\hat{q}; b_t, y_t, b_{t+1})$ is equal to one is intuitive. A probability distribution that places a probability equal to one on $\underline{q}(b_t, y_t, b_{t+1})$ is an equilibrium consistent distribution. For this distribution $\Pr_Q(q \leq \hat{q})$ is going to be equal to one. Thus, the max $\Pr_Q(q \leq \hat{q})$ over the set of equilibrium consistent distributions is equal to one. The case in which $\hat{q} < \underline{q}(b_t, y_t, b_{t+1})$ is not that simple, though. Proposition 2 finds the maximum ex-ante probability (before ζ_t is realized) of observing a price q_t, lower than $\hat{q}$, and it is less than one. To relax the IC constraint for the government, condition (3.2), as much as possible, we do the following: we consider distributions that are binary and assign prices $\{\hat{q}, \overline{q}\}$, and when $\overline{q}$ is realized assign the best continuation equilibria and when $\hat{q}$ is realized assign the best ex-post continuation equilibrium, $\overline{v}(y_t, b_{t+1}, \hat{q})$. The expected value for the government under this distribution, that we label $\underline{Q}(\hat{q}; \cdot)$, needs to be as good as defaulting. When we equalize the value of issuing debt with the distribution $\underline{Q}(\hat{q}; \cdot)$ to the value of defaulting, it specifies an equation for $\underline{Q}(\hat{q}; b_t, y_t, b_{t+1})$, that is precisely given by (3.5).

Note that if the income realization is such that $\overline{V}^{nd}(b_t, y_t) = V^d(y_t)$ (i.e., under the best continuation equilibrium, the government is indifferent between defaulting or not,

correlating devices.

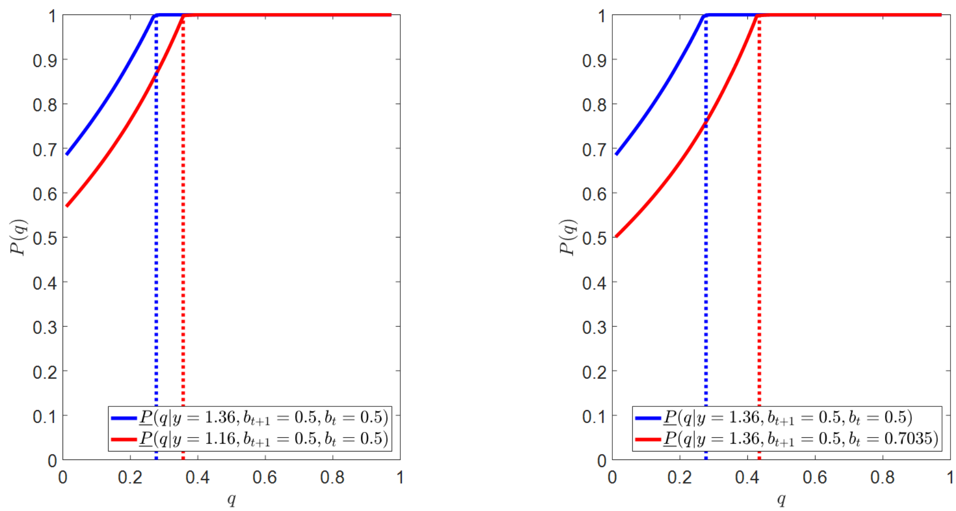

Figure 4: This figure plots $\underline{Q}(q)$ for different levels of output for our main calibrated parameters. The left panel fixes $b_{t+1}$and b_t and shows the comparative statistics with respect to y_t. The right panel fixes y_t and shows the comparative statistics with respect to b_t.

and still does not default), then $\underline{Q}(\hat{q}; b_t, y_t, b_{t+1}) = 0$ for any $\hat{q} < \underline{q}(b_t, y_t, b_{t+1}) = \overline{q}(y_t, b_{t+1})$. The idea is that for these income levels, only $q = \overline{q}(y_t, b_{t+1})$ is an equilibrium consistent price, and the only distribution that is equilibrium consistent places probability one on that price. Note also that $\underline{Q}$ is a cumulative distribution function for q: it is a non-increasing, right-continuous function with a range of $[0, 1]$; hence it implicitly defines a probability measure for debt prices.

Figure 4 presents the function for the maximum probability of low prices, $\underline{Q}(\hat{q}; b_t, y_t, b_{t+1})$, for different states (b_t, y_t, b_{t+1}). In the left panel, the two distributions differ on the income realization under which the government repaid its debt. Lets start with the blue line: the government repaid debt under an income realization (y_t) of 1.36, repaid 0.5 units of debt (b_t), and issued 0.5 units (b_{t+1}). $\underline{Q}(0)$ is approximately 0.7; in other words, the maximum probability of obtaining a price of zero is approximately 0.7. Any distribution where the probability of a price of zero is higher than 0.7, after the history $(b_t, y_t, b_{t+1}) = (0.50, 1.36, 0.50)$, is not equilibrium consistent because it violates the IC constraint of the government. Second, note that as the price $\hat{q}$ increases, $\underline{Q}(\hat{q}; b_t, y_t, b_{t+1})$ also increases: the government is willing to accept a higher probability of obtaining low prices (lower than $\hat{q}$), because these prices are not that low. Third, as we should expect, given our previous discussion, the function $\underline{Q}(\hat{q}; b_t, y_t, b_{t+1})$ reaches one at a price equal to $\underline{q}(b_t, y_t, b_{t+1}) \mid_{(b_t, y_t, b_{t+1})=(0.50, 1.36, 0.50)}$. Fourth, note that the function $\underline{Q}(\hat{q})$ shifts if the government repays its debt under poor economic conditions (these conditions imply a lower spot utility); for example, $\underline{Q}(0)$ is approximately 0.55 instead of 0.7, if income is 1.16 instead of 1.36, which is what one would expect in order not to violate the incentive compatibility constraint, condition (3.2). Finally, the right hand side of the panel shows the comparative statistics with respect to how much debt is repaid.

Bounding Expectations. One application that is of particular interest is bounding the moments of distributions across all equilibria. We start with expected values. The set of equilibrium consistent expected prices is just the set of possible $\int q dQ$ for some $Q \in \mathbb{ECD}(b_t, y_t, b_{t+1})$. Denote this set by $E(b_t, y_t, b_{t+1})$. We will show that this set can be easily characterized, and that this set is related to the prices we studied in the model without sunspots, in subsection 3.2. In fact, the following proposition shows that the set of expected values is identical to the set of equilibrium consistent prices when there are no sunspots.

Proposition 3. *Suppose that history $h_m^t = (h^t, y_t, d_t, b_{t+1})$ is equilibrium consistent. Then the set of expected prices is equal to the set of equilibrium consistent prices without sunspots; i.e.,*

$$E(b_t, y_t, b_{t+1}) = \left[\underline{q}(b_t, y_t, b_{t+1}), \overline{q}(y_t, b_{t+1})\right].$$

Moreover, if $b_{t+1} > 0$, then the minimum expected value is uniquely achieved at the Dirac distribution $\hat{Q}$ that assigns probability one to $q = \underline{q}(b_t, y_t, b_{t+1})$.

Proof. See Appendix A. □

The argument for the proof is based on two facts. First, the monotonicity and the concavity, in q, of the best ex-post continuation value function, $\overline{v}(y_t, b_{t+1}, q)$. Second, that $\underline{q}(\cdot)$ is the minimum price, q, for which $u(y_t - b_t + qb_{t+1}) + \beta\overline{v}(y_t, b_{t+1}, q)$ is equal to $V^d(y_t)$.[19] From the second fact, note that the integrand in the left hand side of condition (3.2) is larger than $V^d(y_t)$ only when q is greater than or equal to $\underline{q}(b_t, y_t, b_{t+1})$. The concavity of $\overline{v}(y_t, b_{t+1}, q)$ and Jensen's inequality then imply that for any distribution $Q \in \mathbb{ECD}(b_t, y_t, b_{t+1})$, $u\left(y_t - b_t + \mathbb{E}_Q(q)\, b_{t+1}\right) + \beta\overline{v}\left(y_t, b_{t+1}, \mathbb{E}_Q(q)\right)$ has to be greater than or equal to $\int \left[u(y_t - b_t + qb_{t+1}) + \beta\overline{v}(y_t, b_{t+1}, q)\right] dQ(q)$. Because Q is an equilibrium consistent distribution, condition (3.2) implies that the latter needs to be greater than or equal to $V^d(y_t)$. Thus, because of the monotonicity of $\overline{v}(y_t, b_{t+1}, q)$, we conclude

[19] The equality at $q = \underline{q}(\cdot)$ follows from the strict monotonicity in q of equilibrium utility, that is given by $u(y_t - b_t + qb_{t+1}) + \beta\overline{v}(y_t, b_{t+1}, q)$. If the inequality were to be strict, then we could find a lower equilibrium consistent price, which contradicts the definition of $\underline{q}(\cdot)$.

that $\mathbb{E}_Q(q_t)$ is greater than $\underline{q}(b_t, y_t, b_{t+1})$. The fact that $\mathbb{E}_Q(q_t)$ is less than or equal to $\overline{q}(y_t, b_{t+1})$, is immediate.

Remark 1. Proposition 3 provides testable implications of equilibria. These implications extend the restrictions derived in the work of Jovanovic (1989) and Pakes et al. (2015). The bounds that we just derived yield moment inequalities; in particular, for every history h_m^{t+1} it holds that $\mathbb{E}_{q_t}\left[q_t \mid h_m^t\right] \in \left[\underline{q}(b_t, y_t, b_{t+1}), \overline{q}(y_t, b_{t+1})\right]$. Aided by these moment inequalities, one could, in principle, perform estimation of the structural set of parameters as in Chernozhukov et al. (2007) and Galichon and Henry (2011).

Bounding Variances. Next, we characterize bounds over variances. The importance of this application comes not only from the fact that we can obtain dynamic implications from equilibria; we can also know, ex-ante, how much volatility the model can generate. Note that without any a-priori knowledge this can be a daunting task. Which equilibrium will yield the highest variance? In the next proposition, we can pin down how much variance the model can generate, without trying every possible equilibrium. Take any $Q \in \mathbb{ECD}(h_m^t)$ with $\mathbb{E}_Q(q_t) = \mu$. Denote by $S\left(h_m^t, \mu\right)$ the set of variances of these distributions.

Proposition 4. *Suppose that history $h_m^t = \left(h^t, y_t, d_t, b_{t+1}\right)$ is equilibrium consistent. Define $q^* := \left[1 - \underline{Q}(0)\right] \times \overline{q}(y_t, b_{t+1})$. If $Q \in \mathbb{ECD}(h_m^t)$ and $\mathbb{E}_Q(q_t) = \mu$; then, $S\left(h_m^t, \mu\right) = \left[0, \overline{\mathbb{V}ar}\left(h_m^t, \mu\right)\right]$ where $\overline{\mathbb{V}ar}\left(h_m^t, \mu\right)$ is defined as:*

- *If $\mu \geq q^*$, then $\overline{\mathbb{V}ar}\left(h_m^t, \mu\right) = \mu\left(\overline{q} - \mu\right)$.*
- *If $\underline{q}(b_t, y_t, b_{t+1}) \leq \mu < q^*$ then $\overline{\mathbb{V}ar}\left(h_m^t, \mu\right) = \mu\left(\overline{q} + q_\mu - \mu\right) - q_\mu \overline{q}$, where q_μ is the unique solution to the equation $\underline{Q}(q_\mu)\, q_\mu + \left(1 - \underline{Q}(q_\mu)\right)\overline{q} = \mu$ and $\underline{Q}(q)$ is defined in Proposition 2.*

Proof. See Appendix A. □

The idea of the proof is as follows. We know that any price distribution with sunspots lies in the interval $[0, \overline{q}(y_t, b_{t+1})]$. We start from the observation that the maximum variance is achieved with a binary distribution. For the first case, we show that the no default incentive constraint (3.2) is not binding if the expected prices are high enough; i.e., if $\mu \geq q^*$. Then, the volatility of the candidate distribution (that has a mean μ, and is binary over $\{0, \overline{q}\}$), is given by $\overline{\mathbb{V}ar}\left(h_m^t, \mu\right) = \mu\left(\overline{q} - \mu\right)$. For the second case, when $\mu < q^*$, the incentive constraint for no-default starts to be binding. The maximum variance is still achieved by a binary distribution, but this binding constraint restricts how low the price

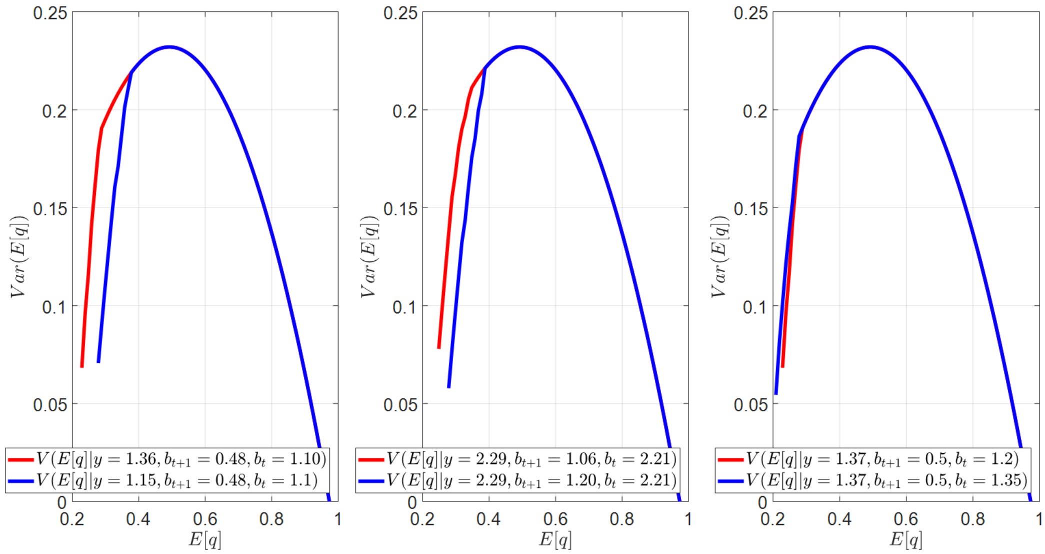

Figure 5: This figure plots $\overline{\mathbb{V}ar}\left(h_m^t, \mu\right)$ for for different levels output and for our main calibrated parameters. The left panel fixes $b_{t+1}$and b_t and perform comparative statistics with respect to y_t. The middle panel fixes $y_t$and b_t and perform comparative statistics with respect to b_{t+1}.The right panel fixes y_t and perform comparative statistics with respect to b_t.

can be in the bad state. Thus, we fix q_μ such that $\Pr\left(q_\mu\right) q_\mu + \left(1 - \Pr\left(q_\mu\right)\right) \overline{q}$ is equal to μ for some probability $\Pr(q_\mu)$. In addition, we choose $\Pr(q_\mu)$ so that the incentive constraint (3.2) is binding for the candidate distribution. This probability is exactly $\underline{Q}\left(q_\mu\right)$. This is intuitive, because will make the probability of the low value as high as possible, maximizing the variance.

Figure 5 presents the bounds of the variances for the equilibrium consistent distributions given an expected value for prices. Each one of the panels and each of the two cases in each panel are different because they display different values of (b_t, y_t, b_{t+1}). First, it is clear that in the three panels, the frontier of the mean and variance has kinks. All these kinks occur when the expected price is equal to q^*. Second, note that in all of the panels both curves are the same up to the kink of the blue line. This result occurs because q^* is a function of (b_t, y_t, b_{t+1}), which marks the kink for each one of the curves. If the expectation of prices, $\mathbb{E}(q)$, is higher than the maximum of both q^* (that is a function of the history), then the variances are identical and given by $\mu\left(\overline{q} - \mu\right)$.[20] In the right panel, the red line falls faster than the red line, because for the blue line the debt repayment is larger ($b_t = 1.35$ and $b_t = 1.2$, respectively); thus, for a given mean the variance needs to

[20]It is worth noting that for values of $\mathbb{E}(q)$ that are higher than q^*, the blue and red lines do not need to coincide. The reason why they coincide is because $\overline{q}(y_t, b_{t+1})$ is flat for both variables in the range of (y_t, b_{t+1}) in the plots.

be smaller. Alternatively, in the middle panel the blue line falls faster. Because more debt is issued in the history that corresponds to the red line, for a given mean, the government tolerates higher variances of prices, without violating condition (3.2).

A General Characterization of Moments. As a corollary of our main result, we formulate a a simple linear program that characterizes all non-centered moments. Note that the moment generating function of prices (which are a random variable) $f(q)$ is given by

$$M_{f(q)}(t) := \mathbb{E}\left(e^{tf(q)}\right),$$

for $t \in \mathbb{R}$. Recall also that the moment generating function of the random variable $f(q)$ pins down all the non centered moments (a standard result in mathematical statistics); in particular

$$\mathbb{E}(f(q^n)) = \frac{d^n}{dt^n}\left(M_{f(q)}(t)\right)|_{t=0}.$$

Therefore, we can characterize the maximum and minimum of the set of moments as a solution of linear programming problem. We state the result, the idea for the proof follow from the previous discussion.

Proposition 5. *Suppose that h_m^t is and equilibrium consistent history. Then, the maximum n-th non centered moment solves the following linear program:*

$$\overline{\mathbb{E}}(f(q^n), h_m^t) := \max_Q \frac{d^n}{dt^n}\left(\mathbb{E}^Q\left(e^{tf(q)}\right)\right)|_{t=0}$$

subject to (3.1) and (3.2).

Proof. Omitted. □

The proof is immediate and follows from the previous discussion. The result for the minimum non centered moment is analogous when we replace the max operator with the min operator. Proposition 5 extends the logic of Propositions 4, 3, and 2. Note that this is a linear programming problem because we can interchange the expectation and the derivative.

Comparative Statics and Stochastic Dominance. We close this subsection by providing the comparative statics over the set of distributions, $\mathbb{ECD}(b_t, y_t, b_{t+1})$.

Corollary 2. *(a) The set of equilibrium price distributions $\mathbb{ECD}(b_t, y_t, b_{t+1})$ is non-increasing (in a set order sense) with respect to b_t and if income is i.i.d, it is non-decreasing in y_t. (b)*

Suppose that $Q \in \mathbb{ECD}(b_t, y_t, b_{t+1})$ and Q' is a probability distribution for equilibrium prices; i.e. $Q' \in \Delta([0, \overline{q}(y_t, b_{t+1})])$. If Q' first order stochastically dominates (FOSD) Q, then $Q' \in \mathbb{ECD}(b_t, y_t, b_{t+1})$. (c) $\underline{Q} \notin \mathbb{ECD}(b_t, y_t, b_{t+1})$. Furthermore, for every $Q \in \mathbb{ECD}(b_t, y_t, b_{t+1})$ it holds that Q FOSD $\underline{Q}$, and if Q' is some other lower bound, then $\underline{Q}$ FOSD Q'.

Proof. See Appendix A. □

The idea of the argument follows. First, the intuition of the first part of these comparative statistics, again, stems from the revealed preference argument. If the government repaid a larger amount of debt, then the distribution of the prices that they would expect needs to shift towards higher prices. If the set does not change, then there will be a distribution that will be inconsistent with equilibrium because it will violate condition (3.2). Second, the proposition shows that once a distribution is consistent with equilibrium, any distribution that FOSD this distribution will be an equilibrium consistent distribution. This is intuitive: higher prices lead to both higher consumption and higher continuation equilibrium values for the government since both are weakly increasing in the debt price q_t. Finally, by its own definition, $\underline{Q}$ is the infimum over all possible distributions in $\mathbb{ECD}$. In addition, the fact that $\underline{Q} \notin \mathbb{ECD}(b_t, y_t, b_{t+1})$ follows immediately from the fact that the support of $\underline{Q}$ is $\left[0, \underline{q}(b_t, y_t, b_{t+1})\right]$.

4 A General Dynamic Policy Game

In this section we show that the main result that we proved Section 3, Proposition 1 extends to a more general class of policy games and do not rely on the specific model studied in these sections. This should not be surprising. The main economic argument for Proposition 1 follows from revealed preference: what the government leaves on the table provides bounds on the expectation it had regarding future play. These bounds place restrictions over outcomes or over distributions. Therefore, in this section we do two things. First, we propose a general model of a dynamic policy game in the spirit of Stokey et al. (1989).[21] Second, for this more general setup we provide of Proposition 1.

[21] To keep notation simple and the exposition more concrete, we will focus on games in which the short run players form an expectation regarding next period policy. There is a large class of models that share this timing. For sovereign debt, one class follows Eaton and Gersovitz (1981). For monetary policy, one class is the New Keynesian model as in Benigno and Woodford (2003). There are policy games that focus on alternative timings, though. In particular, there is a class of games in which the decision of the long-lived player and the short-lived players occurs sequentially, but in the same period. This timing has been used mainly for monetary policy (for example, in the seminal contribution of Barro and Gordon, 1983, but see also, for example, Obstfeld et al., 1996), and capital taxation (see for example Phelan, 2006 and Chari and Kehoe, 1990). Our results can be extended to incorporate these alternative timings.

4.1 Setup

We will follow the notation in Stokey et al. (1989). There are two types players: an infinitely long lived player (government) and short lived agents (market) that set expectations according to a particular rule. In each period t agents play an extensive form stage game with 5 sub periods $(t, \tau_i)_{i\in\{1,5\}}$. The payoff relevant states are an exogenous random shock y_t, and an endogenous state variable b_t. The timeline of the stage game follows:

- $\tau = \tau_1$: A publicly observable random variable $y_t \in Y \subseteq \mathbb{R}^l$ is realized, that follows a (controlled) Markov process: $y_t \sim f(y \mid y_{t-1}, b_t)$.[22]

- $\tau = \tau_2$: The long-lived player (government) chooses a control $d_t \in D \subseteq \mathbb{R}^d$ and a next period state variable $b_{t+1} \in B \subset \mathbb{R}^b$ (where both D and B are compact sets). We say that (d_t, b_{t+1}) is feasible if $(d_t, b_{t+1}) \in \Gamma(b_t, y_t)$, where $\Gamma : B \times Y \rightrightarrows D \times B$ is a non-empty, compact valued and continuous correspondence.

- $\tau = \tau_3$: A sunspot variable ζ_t is realized and distributed according to $\zeta_t \sim U[0, 1]$.

- $\tau = \tau_4$: The agents determine their expectations about future play. This process is modeled in reduced form, with the market choosing $q_t \in \mathbb{R}^k$ to satisfy:

$$q_t = \mathbb{E}_t \left\{ \sum_{s=t}^{\infty} \delta^{s-t} T\left(b_{s+1}, y_{s+1}, d_{s+1}, b_{s+2}\right) \right\}$$

  where $\delta \in (0, 1)$ and $T : B \times Y \times D \times B \to \mathbb{R}^k$ is a continuous and bounded function. The expectation is taken over future shocks $\{y_{t+s}\}_{s=1}^{\infty}$ knowing the strategy profile of the long lived player.

- $\tau = \tau_5$: the payoffs for the long lived player are realized and given by a continuous utility function $u(b_t, y_t, d_t, b_{t+1}, q_t)$. Lifetime utility is then given by

$$V_0 := \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u\left(b_t, y_t, d_t, b_{t+1}, q_t\right) \right\},$$

  where $\beta \in (0, 1)$.

Example 1. This example is exactly the one studied in Section 2. y_t is national income, $b_t \geq 0$ is the outstanding public debt to be repaid, $d_t \in \{0, 1\}$ is the default decision

[22] Sometimes, we say that y *includes a sunspot* if $\exists \{y_t^*, z_t\}$ such that (1) $y_t^* \perp z_t$ for all t, (2) y_t^* is a controlled Markov process; i.e. $y_t^* \sim g\left(y_t^* \mid y_{t-1}^*, b_t\right)$ and (3) $z_t \sim_{i.i.d}$ Uniform $[0, 1]$.

and $q_t = \mathbb{E}_t\left[\frac{1-d_{t+1}}{1+r}\right]$ is the risk neutral price set by lenders in equilibrium. Flow utility is given by $u\left(b_t, y_t, d_t, b_{t+1}, q_t\right) = \left(1-d_t\right)u\left(y_t - b_t + q_t b_{t+1}\right) + d_t u\left(y_t\right)$, assuming that when the government defaults on its debt, it gets to consume its income and is banned forever from international financial markets. Note that the feasibility correspondence is given by $\Gamma(y_t, b_t, q_t) = y_t - b_t + q_t b_{t+1} \geq 0$.[23]

Example 2. Our framework also incorporates New Keynesian (NK) models of monetary policy with no endogenous state; see for example Benigno and Woodford (2003) and more recently Waki et al. (2018). In the case of the NK model the control is $d_t = \pi_t$ where π_t is inflation. Agents set inflation expectations to match future inflation, as $q_t := \pi_t^e = \mathbb{E}_t\left(\pi_{t+1}\right)$. Inflation and output are related according to a forward looking Phillips curve $x_t = \pi_t - \beta\pi_t^e + \epsilon_t$, where x_t is the output gap and ϵ_t is a supply shock. In addition, let π_t^* be a random variable that gives the optimal natural level of inflation (absent an inflation gap). The random shocks are then $y_t = \pi_t^*$, and the government is assumed to minimize the loss function:

$$\mathcal{L}\left(\pi, \pi^e, y_t\right) = \frac{\chi}{2}\left(\pi_t - \beta\pi_t^e\right)^2 + \frac{1}{2}\left(\pi_t - y_t\right)^2,$$

where the first term in the loss function is the output gap. In this example, the feasibility constraint represents the fact that π_t needs to be bounded.

Histories, Equilibrium and Equilibrium Consistency. The notation in this section follows the one used in Section 2. Recall that a *history* is a vector $h^t = (h_0, h_1, ..., h_{t-1})$, where $h_t = (y_t, d_t, b_{t+1}, q_t)$ is the description of the outcome of the stage game at time t. A partial history is an initial history h^t concatenated with some subset of the stage game at period t. The set of all partial histories (initial and partial) is denoted by $\mathcal{H}$, and $\mathcal{H}_g \subset \mathcal{H}$ represent the histories where the government has to choose (d_t, b_{t+1}); i.e., $\left(h^t, y_t\right)$. Likewise, $\mathcal{H}_m \subset \mathcal{H}$ is the set of partial histories where the "market" sets its expectations; i.e., $h_m^t = \left(h^t, y_t, d_t, b_{t+1}\right)$. A *strategy* for the *government* is a function $\sigma_g\left(h^t, y_t\right) = \left(d_t^{\sigma_g}, b_{t+1}^{\sigma_g}\right)$ for all histories, and a strategy for the *market* is a pricing function $q_m\left(h^t, y_t, d_t, b_{t+1}, \zeta_t\right)$. The payoff for the government of a particular (feasible) strategy $\left(\sigma_g, \sigma_m\right)$, after a particular history $\left(h^t, y_t\right)$ is given by:

$$V\left(\sigma \mid h^t, y_t\right) := \mathbb{E}_t\left\{\sum_{t=s}^{\infty}\beta^{t-s}u\left(b_t^{\sigma_g}, y_t, d_t^{\sigma_g}, b_{t+1}^{\sigma_g}, q_t^{\sigma_m}\right)\right\}.$$

[23] As we comment in the Online Appendix, because the market chooses after the government it can be the case that this constraint is ex-post "violated". In that case, the government has a technology available to generate resources such that the budget constraint holds; in this case the government obtains utility of $-\infty$.

A strategy profile $\sigma = (\sigma_g, \sigma_m)$ is a *Subgame Perfect Equilibrium* (SPE) of the game if:

a. $V(\sigma \mid h^t, y_t) \geq V\left(\sigma'_g, q_m \mid h^t, y_t\right)$ for all $(h^t, y_t), \sigma'_g \in \Sigma_g$;

b. $q_m(h^t, y_t, d_t, b_{t+1}, \zeta_t) = \mathbb{E}_t\left\{\sum_{s=t}^{\infty} \delta^{s-t} T(b_{s+1}, y_{s+1}, d_{s+1}, b_{s+2})\right\}$ where the policies $(b_{s+1}$ $d_{s+1}, b_{s+2})$ are generated by σ.

We denote it by $\sigma \in \Sigma^*$. The methodology we developed in section 3 derived restrictions imposed by equilibrium histories over continuation equilibria. We focused on a particular dynamic policy game that followed Eaton and Gersovitz (1981). In this section we follow similar steps for the general model that we just described. Given a SPE profile $\sigma = (\sigma_g, q_m)$, we define its *equilibrium path* $x = x(\sigma)$ as a sequence of measurable functions $x = \left(d_t(y^t, \zeta^{t-1}), b_{t+1}(y^t, \zeta^t), q_t(y^t, \zeta^t)\right)_{t \in \mathbb{N}}$ that are generated by following the profile σ. A history h is *equilibrium consistent* if and only if it is on the equilibrium path $x = x(\sigma)$, for some subgame perfect equilibrium $\sigma \in \Sigma^*$.

Equilibrium Values. As we did in Section 2, it is useful to define the best ex-post continuation payoff. Also, we define the set of equilibrium payoffs and and the worst equilibrium payoff. Denote as $\mathcal{E}(y_-, b)$ as the set of equilibrium payoffs. Formally, $\mathcal{E}(y_-, b)$ is defined as:

$$\mathcal{E}(y_-, b) := \left\{ (q, v) \in \mathbb{R}^k \times \mathbb{R} : \exists \sigma \in \Sigma^*(y_-, b) \text{ with } \begin{array}{c} v = V(\sigma \mid h_0 = (y_-, b)) \\ q = \mathbb{E}_0\left\{\sum_{t=0}^{\infty} \delta^t T(b_{t+1}, y_{t+1}, d_{t+1}, b_{t+2}) \mid y_-, b\right\} \end{array} \right\}$$

and let $\mathcal{Q}(y_-, b) \subseteq \mathbb{R}^k$ be its projection over q. We can characterize $\mathcal{E}(y_-, b)$ using the concept of self-generation and enforceability in Abreu (1988); Abreu et al. (1990) and Atkeson (1991). It can be shown that if y is non-atomic and u is concave in q (for example, risk aversion of the long lived player), then $\mathcal{E}(y_-, b)$ is compact and convex valued. This is satisfied by both examples discussed above. For a simpler exposition, for the rest of this section we focus on this case.[24]

Best Continuation Value. We continue with the *best value function* and the *max-min value*. The best value function gives the maximum equilibrium value for the long lived

[24]Why is the exposition simpler? When this is not the case, all the propositions in this section remain valid, but we need to define the functions $\overline{v}$, the best ex-post continuation payoff and $\underline{U}$, the worst equilibrium payoff, over the correspondence $\mathcal{E}^s(y_-, b)$ instead. These two functions are defined as $\overline{v}^s(y, b, q) = \max\{v : (q, v) \in \mathcal{E}^s(y, b)\}$ and $\underline{U}^s(y, b) := \max_{(d, b') \in \Gamma(b, y)} \min_{(q, v) \in \mathcal{E}^s(y, b)} u(b, y, d, b', q) + \beta v$.

player, if $q_t = q_-$ is realized; i.e.,

$$\overline{v}(y_-, b, q_-) := \max_{v \in \mathbb{R}} v \tag{4.1}$$

$$\text{s.t. } (q_-, v) \in \mathcal{E}(y_-, b).$$

By following steps that are similar to the ones used in the Appendix, Section C, we can also show that if $\mathcal{E}(y_-, b)$ is convex valued and $u(\cdot)$ is concave in q, then $\overline{v}(y_-, b, q_-)$ is also concave in q. The *max-min value* is the worst possible value that the long lived player can obtain in any SPE , going forward. Formally,

$$\underline{U}(y, b) := \max_{(d,b') \in \Gamma(b,y)} \left\{ \min_{(q,v) \in \mathcal{E}(y,b')} u(b, y, d, b', q) + \beta v \right\}. \tag{4.2}$$

In the sovereign debt model $\underline{U}(y, b)$ is equal to $V^d(y)$.[25]

Equilibrium Consistency. We present a generalization of the main result presented in Section 3, Proposition 1, for the general model that we just introduced.

Proposition 6. *Suppose that h_m^t is an equilibrium consistent history. Then, Q_t is an equilibrium consistent distribution if and only if: (a) SPE prices; i.e.*

$$Q_t \in \Delta[\mathcal{Q}(y_t, b_{t+1})]$$

(b) *incentive compatibility for the long lived player:*

$$\int_{\hat{q} \in \mathcal{Q}(y_t, b_{t+1})} [u(b_t, y_t, d_t, b_{t+1}, \hat{q}) + \beta \overline{v}(y_t, b_{t+1}, \hat{q})] \, dQ_t(\hat{q}) \geq \underline{U}(y_t, b_t).$$

Proof. Omitted. □

The proof of Proposition 6 follows closely the steps of the proof of Proposition 1. We discuss the argument in the Online Appendix Section A.6. Proposition 6 generalizes Proposition 1 for the case in which $\mathcal{E}$ is convex valued and u is concave in q. Again, the first requirement, $Q_t \in \Delta[\mathcal{Q}(y_t, b_{t+1})]$ is asking for a distribution to be a probability distribution over equilibrium prices. As in Section 3, we can use the results in Proposition 6 to obtain observable implications over prices. For example, we

[25]There are several papers that develop the techniques to solve for the set of equilibrium payoffs following the seminal contribution of Judd et al. (2003). Following Waki et al. (2018), it can be shown that $\overline{v}(y_-, b, q_-)$ can be expressed as the unique fixed point of a contraction mapping, given $\underline{U}(y, b)$.

can again obtain bounds over expectations. Define now a set of equilibrium consistent price distributions $\mathbb{ECD}(b_t, y_t, d_t, b_{t+1})$. Because the IC for the government is a linear inequality on measures Q_t, under the assumptions of Proposition 6 the function $g(\hat{q} \mid h_t) := u(b_t, y_t, d_t, b_{t+1}, \hat{q}) + \beta\overline{v}(y_t, b_{t+1}, \hat{q}) - \underline{U}(y_t, b_t)$ is concave in $\hat{q}$ as well. Therefore, as in the sovereign debt model, it can be shown that the set of expected prices $E(b_t, y_t, b_{t+1})$, that is defined as the values $\int \hat{q} dP(\hat{q})$ such that $Q \in \mathbb{ECD}(b_t, y_t, d_t, b_{t+1})$, is equal to the set of equilibrium consistent prices without sunspots; i.e. $E(b_t, y_t, d_t, b_{t+1}) = \text{proj}_Q \mathbb{ECD}(b_t, y_t, d_t, b_{t+1})$.

Remark 2. We can also extend this result for a restricted set of equilibria $\hat{\Sigma} \subseteq \Sigma^{SPE}$, in particular those for which there exist a self-generating correspondence[26] $W(y_-, b) \subseteq \mathcal{E}(y_-, b)$ such that $\sigma \in \hat{\Sigma}$ if and only if $(V(\sigma \mid h), q_m(h)) \in W(y_, , b)$ for all histories h. This is the case when we restrict "punishments" (i.e. penalties for deviating from the equilibrium strategy) by a function $G(y_, b)$. In sovereign debt models, the literature studies equilibria where punishment is restricted to be a random (but finite) number of periods on autarky. In Appendix D we show how we can obtain the equivalent $\overline{v}(y_-, b, q)$ from this constraint on punishments,

4.2 New Keynesian Model

In this subsection we perform an additional application focusing in the New Keynesian Model as in Benigno and Woodford (2003) without sunspots. The sequence of events is as follows. First, the shock $\pi_t^* = y_t$ is realized (inflation target), where y_t is an i.i.d. random variable with zero mean and volatility σ_y. Then, the monetary authority chooses $d_t = \pi_t$. Finally, markets set expected inflation $\pi_t^e = \mathbb{E}_t(\pi_{t+1})$. The relationship between inflation and output is

$$\pi_t = \kappa x_t + \beta \pi_t^e,$$

and the Loss Function for the monetary authority is

$$\mathcal{L}(x_t, \pi_t) = \frac{\chi}{2} x_t^2 + \frac{1}{2}(\pi_t - y_t)^2.$$

Equilibrium. We study the set of equilibria where the punishment is bounded by the stationary equilibrium of this game.[27] In the online Appendix we show that since there

[26] See Appendix C.2

[27] Monetary models feature wide equilibrium multiplicity, and in the worst equilibrium the value of money is zero, with potentially unbounded punishments (and hence any behavior could be supported in an equilibrium path). There are two solutions. One possibility is to assume a bounded set for inflation

are no payoff relevant state variable and shocks are assumed to be i.i.d, the value for the *worst stationary equilibrium* payoff $\underline{U}^s(y,b) = \underline{U}^s(y)$ is given by:

$$\underline{U}^s(y) = -\frac{\hat{\chi}}{2(1+\hat{\chi})}\left(y^2 + \beta\frac{\sigma_y^2}{1-\beta}\right). \tag{4.3}$$

In the online Appendix, following Waki et al. (2018), we show that: (a) the best continuation value function solves

$$\overline{v}(\pi^e) = \max_{\pi(\cdot),x(\cdot),\pi^e_+(\cdot)} \mathbb{E}_y\left\{-\mathcal{L}(\pi(y),\pi^e(y),y) + \beta\overline{v}(\pi^e_+(y))\right\} \text{ s.t } \mathbb{E}_y(\pi(y)) = \pi^e$$

where $\mathcal{L}(\pi,\pi^e,y) = \frac{\hat{\chi}}{2}(\pi-\beta\pi^e)^2 + \frac{1}{2}(\pi-y)^2$ and $\pi^e_+(\cdot)$ denotes next period expected inflation; and, (b) that $\overline{v}(\cdot)$ is in fact a quadratic, concave function of expected inflation, and given by

$$\overline{v}(\pi^e) = -\frac{\delta}{2}(\pi^e)^2 - \gamma, \tag{4.4}$$

which are two positive constants that are closed form expressions of the parameters.

Equilibrium Consistency. Using (4.3) and (4.4) we can readily apply Proposition 6 to show that given a shock y_t a pair $\left(\pi_t,\pi^e_{t+1}\right)$ *is equilibrium consistent* if and only if

$$-\mathcal{L}\left(\pi_t,\pi^e_{t+1},y_t\right) - \beta\overline{v}\left(\pi^e_{t+1}\right) \geq \underline{U}^s(y_t). \tag{4.5}$$

Plugging (4.3) and (4.4) into (4.5), in the Appendix we show that, given y_t, the set of equilibrium consistent levels of inflation and expected inflation $\left(\pi_t,\pi^e_{t+1}\right)$ solves:

$$(1+\hat{\chi})\pi_t^2 - 2\beta\hat{\chi}\pi_t\pi^e_{t+1} + \beta(\delta+\beta\hat{\chi})\left(\pi^e_{t+1}\right)^2 - 2y_t\pi_t + \frac{y_t^2}{1+\hat{\chi}} - \eta \leq 0 \tag{4.6}$$

where $\hat{\chi} = \frac{\chi}{\kappa^2}$ and $\eta = \left(\frac{1}{1+\hat{\chi}} - \delta\varphi\right)\frac{\hat{\chi}\beta\sigma_y^2}{1-\beta}$. This inequality represents the area inside an ellipse, and can therefore be represented as a range of feasible contemporaneous inflation levels $\pi_t \in [\underline{\pi}(y_t), \overline{\pi}(y_t)]$ and bounds on expected inflation $\pi^e_{t+1} \in [\underline{\pi}^e(y_t,\pi_t), \overline{\pi}^e(y_t,\pi_t)]$.[28]

levels $\pi \in \left[\underline{\Pi},\overline{\Pi}\right]$ and output gap $x \in \left[\underline{X},\overline{X}\right]$ as in Athey et al. (2005) and Waki et al. (2018). Another possibility is to restrict out of equilibrium payoffs (as mentioned in Remark 2) as a way of bounding the off path possibilities in the game. To illustrate the methodology, in this exercise we focus on the case where payoffs are bounded by the (self sustained) stationary equilibrium of this game, where the monetary authority has no ability to influence market beliefs (on or off path).

[28]See Online Appendix E.3 for a derivation.

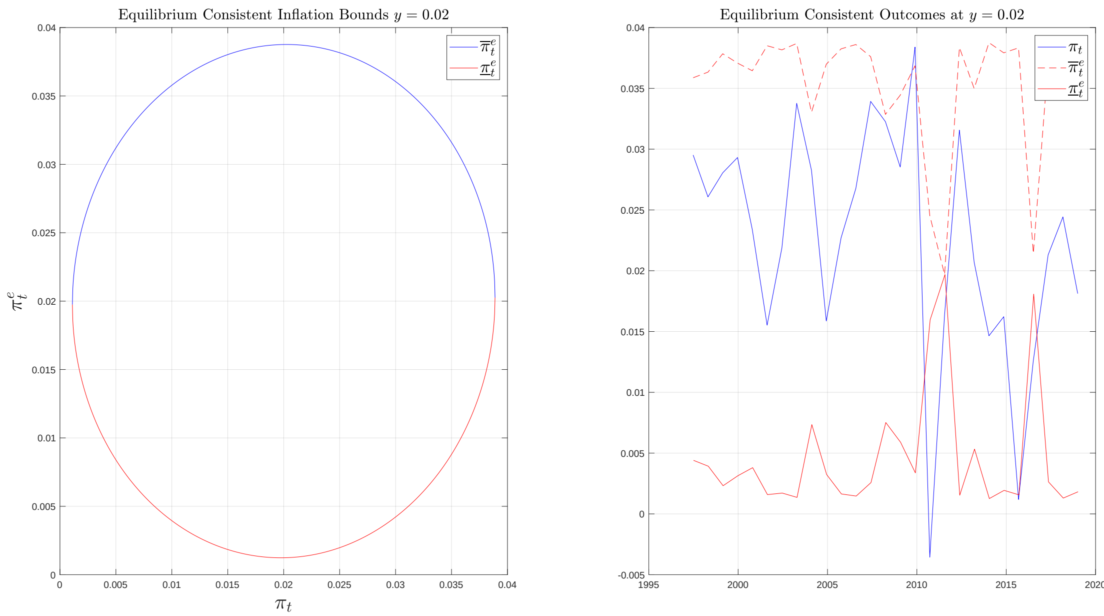

Figure 6: The left panel depicts the equilibrium consistent levels of expected inflation $[\underline{\pi}^e(y_t, \pi_t), \overline{\pi}^e(y_t, \pi_t)]$ given $y_t = 0.02$ for different values of π_t. The parameters are $\beta = 0.995$, $\chi = 4$ and $\kappa = 0.17$ following Galí (2015). The right hand side panel computes the bounds $[\underline{\pi}^e(y_t, \pi_t), \overline{\pi}^e(y_t, \pi_t)]$ by feeding in a value of $y_t = 0.02$ and the realized inflation (year to year change) level π_t for the US in the period 1995 to 2020. The blue line depicts the inflation level π_t over time. The red and red-dashed lines depict the bounds on expected inflation π^e_{t+1}.

Illustration. In Figure 4.2 we show a numerical example for the case with $\mathbb{E}(y) = 0.02$ and $\sigma_y^2 = .01$. We set $\beta = 0.995$, $\chi = 4$ and $\kappa = 0.17$ to standard values in the literature following Galí (2015). In the left of the figure, given $y_t = 0.02$, we plot the bounds on π^e_{t+1} that are characterized by (4.6). Note that when current inflation is 2 percent, then the bounds on expected inflation are approximately between 3.8 percent and 0.2 percent. As current inflation deviates from the target, then the bounds on expected inflation, $[\underline{\pi}^e(y_t, \pi_t), \overline{\pi}^e(y_t, \pi_t)]$, tighten. The intuition is the same one as in the model of sovereign debt: the current losses that the government is bearing today need to be compensated in the future. This compensation comes from a good continuation equilibrium, where inflation goes back to the objective $y_t = 0.02$. The right panel depicts the realized inflation for the annualized quarterly inflation for the US, and the implied bounds on expected inflation over time. Note that inflation has been always consistent with some equilibrium given the parameters.

5 Conclusion

Dynamic policy games have been extensively studied in macroeconomic theory to increase our understanding of how lack of commitment restricts the outcomes that a government can achieve. One of the challenges in studying dynamic policy games is equilibrium multiplicity. Our paper acknowledges and embraces equilibrium multiplicity. For this reason, we focus on obtaining robust predictions: these are predictions that hold across all equilibria; or, in the language of Bergemann and Morris (2018), across every possible information structure.

We obtain robust predictions by characterizing what we term as *equilibrium consistent outcomes*. As we discuss in the text, the basis of our predictions is a revealed preference argument, which is also exploited to obtain the testable implications of equilibria in Jovanovic (1989) and Pakes et al. (2015). The idea of the revealed preference argument is that by taking a particular action, the government obtained some utility; and by doing so, incurred on some opportunity cost. This implied opportunity cost places bounds on what the government can receive in the future. Equilibrium consistency is a general principle. Even though we focus on a model of sovereign debt that follows Eaton and Gersovitz (1981), our results can be generalized to other dynamic policy games, as we show in the last section of the paper.

There are two main directions for further research. First, we think that the predictions we obtain, in particular, the bounds on moments across distributions, provide testable implications of the model that are not sensitive to a particular equilibrium selection mechanism, and thus, can be the basis of estimation procedures robust to equilibrium selection. These estimation procedures can be extensions of the ones in an extensive literature in industrial organization (for example Berry, 1992, Bajari et al., 2007, Aguirregabiria and Mira, 2007) and econometrics (for instance Chernozhukov et al., 2007 and Galichon and Henry, 2011) that recovers structural parameters of interest using moment conditions. However, this link is not immediate. The reason is that one of the crucial assumptions for any econometric estimation procedure is that a version of the law of large numbers holds. For this, we would need to characterize equilibria that on its path meet minimal ergodicity requirements, which is not a straightforward task. Second, and finally, another special feature of our setup is that both the government and the market share common information. Relaxing this assumption would bring our environment closer to the literature on information design, as in Kamenica and Gentzkow (2011), where there is incomplete information. However, our objective would stay the same: obtaining testable implications from the theory when there is asymmetric information. All of these are paths for further research.

References

Abreu, Dilip, "On the Theory of Infinitely Repeated Games with Discounting," *Econometrica*, 1988, *56* (2), pp. 383–396.

_ **, David Pearce, and Ennio Stacchetti**, "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring," *Econometrica*, 1990, *58* (5), pp. 1041–1063.

Aguiar, Mark and Gita Gopinath, "Defaultable debt, interest rates and the current account," *Journal of International Economics*, 2006, *69* (1), 64 – 83.

_ **and Manuel Amador**, "Sovereign Debt: A Review," Working Paper 19388, National Bureau of Economic Research August 2013.

_ **, Satyajit Chatterjee, Harold Cole, and Zachary Stangebye**, "Self-Fulfilling Debt Crises, Revisited: The Art of the Desperate Deal," Technical Report, Working Paper 2020.

Aguirregabiria, Victor and Pedro Mira, "Sequential estimation of dynamic discrete games," *Econometrica*, 2007, *75* (1), 1–53.

Amador, Manuel, "Sovereign Debt and the Tragedy of the Commons," 2013.

Angeletos, George-Marios and Alessandro Pavan, "Selection-free predictions in global games with endogenous information and multiple equilibria," *Theoretical Economics*, 2013, *8* (3), 883–938.

Arellano, Cristina, "Default Risk and Income Fluctuations in Emerging Economies," *The American Economic Review*, 2008, *98* (3), 690–712.

_ **and Yan Bai**, "Renegotiation policies in sovereign defaults," *American Economic Review*, 2014, *104* (5), 94–100.

_ **and** _ , "Fiscal austerity during debt crises," *Economic Theory*, 2017, *64* (4), 657–673.

Athey, Susan, Andrew Atkeson, and Patrick J. Kehoe, "The Optimal Degree of Discretion in Monetary Policy," *Econometrica*, 2005, *73* (5), 1431–1475.

Atkeson, Andrew, "International Lending with Moral Hazard and Risk of Repudiation," *Econometrica*, 1991, *59* (4), pp. 1069–1089.

Auclert, Adrien and Matthew Rognlie, "Unique equilibrium in the Eaton–Gersovitz model of sovereign debt," *Journal of Monetary Economics*, 2016, *84*, 134–146.

Aumann, Robert J., "Correlated Equilibrium as an Expression of Bayesian Rationality," *Econometrica*, 1987, *55* (1), pp. 1–18.

Ayres, Joao, Gaston Navarro, Juan Pablo Nicolini, and Pedro Teles, "Sovereign default: the role of expectations," *Journal of Economic Theory*, 2018, *175*, 803–812.

Bajari, Patrick, C Lanier Benkard, and Jonathan Levin, "Estimating dynamic models of imperfect competition," *Econometrica*, 2007, *75* (5), 1331–1370.

_ , Han Hong, and Stephen P Ryan, "Identification and estimation of a discrete game of complete information," *Econometrica*, 2010, *78* (5), 1529–1568.

Barro, Robert J and David B Gordon, "Rules, discretion and reputation in a model of monetary policy," *Journal of monetary economics*, 1983, *12* (1), 101–121.

Benigno, Pierpaolo and Michael Woodford, "Optimal monetary and fiscal policy: A linear-quadratic approach," *NBER macroeconomics annual*, 2003, *18*, 271–333.

Benoît, Jean-Pierre and Juan Dubra, "Apparent overconfidence," *Econometrica*, 2011, *79* (5), 1591–1625.

Beresteanu, Arie, Ilya Molchanov, and Francesca Molinari, "Sharp identification regions in models with convex moment predictions," *Econometrica*, 2011, *79* (6), 1785–1821.

Bergemann, Dirk and Stephen Morris, "Robust predictions in games with incomplete information," *Econometrica*, 2013, *81* (4), 1251–1308.

_ and _ , "Information design: A unified perspective," *Journal of Economic Literature*, 2018.

_ , Benjamin Brooks, and Stephen Morris, "The limits of price discrimination," *American Economic Review*, 2015, *105* (3), 921–57.

_ , Tibor Heumann, and Stephen Morris, "Information and volatility," *Journal of Economic Theory*, 2015, *158*, 427–465.

Berry, Steven T, "Estimation of a Model of Entry in the Airline Industry," *Econometrica: Journal of the Econometric Society*, 1992, pp. 889–917.

Bianchi, Javier, Juan Carlos Hatchondo, and Leonardo Martinez, "International reserves and rollover risk," *American Economic Review*, 2018, *108* (9), 2629–70.

Bocola, Luigi and Alessandro Dovis, "Self-fulfilling debt crises: A quantitative analysis," *American Economic Review*, 2019, *109* (12), 4343–77.

Bresnahan, Timothy F and Peter C Reiss, "Entry in monopoly market," *The Review of Economic Studies*, 1990, *57* (4), 531–553.

Bugni, Federico A, "Bootstrap inference in partially identified models defined by moment inequalities: Coverage of the identified set," *Econometrica*, 2010, *78* (2), 735–753.

Burks, Stephen V, Jeffrey P Carpenter, Lorenz Goette, and Aldo Rustichini, "Overconfidence and social signalling," *Review of Economic Studies*, 2013, *80* (3), 949–983.

Calvo, Guillermo A, "On the time consistency of optimal policy in a monetary economy," *Econometrica: Journal of the Econometric Society*, 1978, pp. 1411–1428.

Calvo, Guillermo A., "Servicing the Public Debt: The Role of Expectations," *The American Economic Review*, 1988, *78* (4), pp. 647–661.

Chahrour, Ryan and Robert Ulbricht, "Robust predictions for DSGE models with incomplete information," *Available at SSRN 3292550*, 2020.

Chang, Roberto, "Credible Monetary Policy in an Infinite Horizon Model: Recursive Approaches," *Journal of Economic Theory*, 1998, *81* (2), 431 – 461.

_ , "Credible monetary policy in an infinite horizon model: Recursive approaches," *journal of economic theory*, 1998, *81* (2), 431–461.

Chari, V. V. and Patrick J. Kehoe, "Sustainable Plans," *Journal of Political Economy*, 1990, *98* (4), pp. 783–802.

Chari, V.V. and Patrick J. Kehoe, "Sustainable Plans and Debt," *Journal of Economic Theory*, 1993, *61* (2), 230 – 261.

Chatterjee, Satyajit and Burcu Eyigungor, "Maturity, Indebtedness, and Default Risk," *American Economic Review*, 2012, pp. 2674–2699.

_ **and** _ , "A seniority arrangement for sovereign debt," *The American Economic Review*, 2015, *105* (12), 3740–3765.

Chernozhukov, Victor, Han Hong, and Elie Tamer, "Estimation and confidence regions for parameter sets in econometric models 1," *Econometrica*, 2007, *75* (5), 1243–1284.

Ciliberto, Federico and Elie Tamer, "Market structure and multiple equilibria in airline markets," *Econometrica*, 2009, *77* (6), 1791–1828.

Cole, Harold L. and Timothy J. Kehoe, "Self-Fulfilling Debt Crises," *The Review of Economic Studies*, 2000, *67* (1), pp. 91–116.

Corsetti, Giancarlo and Luca Dedola, "The mystery of the printing press: Monetary policy and self-fulfilling debt crises," *Journal of the European Economic Association*, 2016, *14* (6), 1329–1371.

Doval, Laura and Jeff Ely, "Sequential information design," Technical Report, Working Paper 2020.

Dovis, Alessandro, "Efficient Sovereign Default," *Review of Economic Studies*, Forthcoming.

Duffie, Darrell, Piotr Dworczak, and Haoxiang Zhu, "Benchmarks in search markets," *The Journal of Finance*, 2017, *72* (5), 1983–2044.

Eaton, Jonathan and Mark Gersovitz, "Debt with Potential Repudiation: Theoretical and Empirical Analysis," *The Review of Economic Studies*, 1981, *48* (2), pp. 289–309.

Ely, Jeffrey, Alexander Frankel, and Emir Kamenica, "Suspense and surprise," *Journal of Political Economy*, 2015, *123* (1), 215–260.

Farhi, Emmanuel, Christopher Sleet, Ivan Werning, and Sevin Yeltekin, "Non-linear Capital Taxation Without Commitment," *The Review of Economic Studies*, 2012, *79* (4), 1469–1493.

Forges, Françoise, "An approach to communication equilibria," *Econometrica: Journal of the Econometric Society*, 1986, pp. 1375–1385.

Galí, Jordi, *Monetary policy, inflation, and the business cycle: an introduction to the new Keynesian framework and its applications*, Princeton University Press, 2015.

Galichon, Alfred and Marc Henry, "Set identification in models with multiple equilibria," *The Review of Economic Studies*, 2011, *78* (4), 1264–1298.

Gentzkow, Matthew and Emir Kamenica, "Costly persuasion," *American Economic Review*, 2014, *104* (5), 457–62.

__ **and** __ , “Competition in persuasion,” *The Review of Economic Studies*, 2016, *84* (1), 300–322.

Hatchondo, Juan Carlos, Leonardo Martinez, and Cesar Sosa-Padilla, “Debt Dilution and Sovereign Default Risk,” *Journal of Political Economy*, 2016, *124* (5), 1383–1422.

Inostroza, Nicolas and Alessandro Pavan, “Persuasion in global games with application to stress testing,” *Working Paper*, 2020.

Ireland, Peter N, “Sustainable monetary policies,” *Journal of Economic Dynamics and Control*, 1997, *22* (1), 87–108.

Jovanovic, Boyan, “Observable implications of models with multiple equilibria,” *Econometrica: Journal of the Econometric Society*, 1989, pp. 1431–1437.

Judd, Kenneth L, Sevin Yeltekin, and James Conklin, “Computing supergame equilibria,” *Econometrica*, 2003, *71* (4), 1239–1254.

Kamenica, Emir and Matthew Gentzkow, “Bayesian persuasion,” *American Economic Review*, 2011, *101* (6), 2590–2615.

Kolotilin, Anton, Tymofiy Mylovanov, Andriy Zapechelnyuk, and Ming Li, “Persuasion of a privately informed receiver,” *Econometrica*, 2017, *85* (6), 1949–1964.

Kydland, Finn E. and Edward C. Prescott, “Rules Rather than Discretion: The Inconsistency of Optimal Plans,” *Journal of Political Economy*, 1977, *85* (3), pp. 473–492.

Lorenzoni, Guido and Ivan Werning, “Slow Moving Debt Crises,” Technical Report 2018.

Mailath, George J and Larry Samuelson, “Repeated games and reputations: long-run relationships,” *Oxford university press*, 2006.

Myerson, Roger B, “Multistage games with communication,” *Econometrica: Journal of the Econometric Society*, 1986, pp. 323–358.

Obstfeld, Maurice, Kenneth S Rogoff, and Simon Wren-lewis, *Foundations of international macroeconomics*, Vol. 30, MIT press Cambridge, MA, 1996.

Oliveira, Henrique De and Rohit Lamba, “Rationalizing dynamic choices,” *Available at SSRN*, 2019.

Ottonello, Pablo and Diego Perez, “The currency composition of sovereign debt,” *American Economic Journal: Macroeconomics*, 2019, *11* (3), 174–208.

Pakes, Ariel, Jack Porter, Kate Ho, and Joy Ishii, "Moment inequalities and their application," *Econometrica*, 2015, *83* (1), 315–334.

Passadore, Juan and Yu Xu, "Iliquidity in Sovereign Debt Markets," *Working Paper*, 2018.

Paula, Aureo De, "Econometric analysis of games with multiple equilibria," *Annu. Rev. Econ.*, 2013, *5* (1), 107–131.

_ **and Xun Tang**, "Inference of signs of interaction effects in simultaneous games with incomplete information," *Econometrica*, 2012, *80* (1), 143–172.

Phelan, Christopher, "Public trust and government betrayal," *Journal of Economic Theory*, 2006, *130* (1), 27 – 43.

_ **and Ennio Stacchetti**, "Sequential Equilibria in a Ramsey Tax Model," *Econometrica*, 2001, *69* (6), 1491–1518.

Pouzo, Demian and Ignacio Presno, "Sovereign default risk and uncertainty premia," *American Economic Journal: Macroeconomics*, 2016, *8* (3), 230–66.

Roch, Francisco and Harald Uhlig, "The dynamics of sovereign debt crises and bailouts," *Journal of International Economics*, 2018, *114*, 1–13.

Romano, Joseph P and Azeem M Shaikh, "Inference for the identified set in partially identified econometric models," *Econometrica*, 2010, *78* (1), 169–211.

Sánchez, Juan M, Horacio Sapriza, and Emircan Yurdagul, "Sovereign default and maturity choice," *Journal of Monetary Economics*, 2018, *95*, 72–85.

Sleet, Christopher, "On credible monetary policy and private government information," *Journal of Economic Theory*, 2001, *99* (1), 338–376.

Stangebye, Zachary R, "Belief Shocks and Long-Maturity Sovereign Debt," *Working paper*, 2019.

Stokey, Nancy L., "Credible public policy," *Journal of Economic Dynamics and Control*, 1991, *15* (4), 627 – 656.

Stokey, Nancy, Robert Lucas, and Edward Prescott, "Recursive Methods in Economic Dynamics," *Harvard University Press*, 1989.

Sugaya, Takuo and Alexander Wolitzky, "Revelation Principles in Multistage Games," Technical Report, Working Paper 2020.

Tamer, Elie, "Partial identification in econometrics," *Annu. Rev. Econ.*, 2010, *2* (1), 167–195.

Waki, Yuichiro, Richard Dennis, and Ippei Fujiwara, "The optimal degree of monetary discretion in a new Keynesian model with private information," *Theoretical Economics*, 2018, *13* (3), 1319–1367.

Woodford, Michael, *Interest and prices: Foundations of a theory of monetary policy*, princeton university press, 2011.

Appendix to "Robust Predictions in Dynamic Policy Games"

Juan Passadore and Juan Xandri

A Proofs

A.1 Proposition 1

Proof. *Step 1: Necessity.*($\Longrightarrow$). *Step 1.1. Incentive compatibility of no default.* Let $\mathcal{H}(\sigma)$ be the histories on the path of a strategy profile σ. Suppose that there is an equilibrium strategy σ such that $h_m^t \in \mathcal{H}(\sigma)$ and that there is no default so far. The fact that the government optimally decided not to default at period t, implies the following:

$$\int_0^1 \left[u\left(y_t - b_t + q^\sigma\left(h_m^t, \zeta_t\right) b_{t+1}\right) + \beta V^\sigma\left(h_m^t, \zeta_t\right)\right] d\zeta_t \geq u\left(y_t\right) + \beta \mathbb{E}_{y_{t+1}|y_t} V^d(y_{t+1}). \tag{A.1}$$

In other words, the expected payoff of not defaulting needs to be weakly larger than the payoff of defaulting. *Step 1.2. Bounding equilibrium payoffs.* Recall that $\mathcal{E}\left(y_t, b_{t+1}\right)$ is the set of equilibrium payoffs of the game.[29] Since σ is an SPE it holds that for all sunspot realizations $\zeta_t \in [0,1]$:

$$\left(V^\sigma\left(h_m^t, \zeta_t\right), q^\sigma\left(h_m^t, \zeta_t\right)\right) \in \mathcal{E}\left(y_t, b_{t+1}\right).$$

That is, the continuation payoffs for both the government and the market are equilibrium payoffs. This further implies two things:

a. $q^\sigma\left(h_m^t, \zeta_t\right) \in \left[0, \overline{q}\left(y_t, b_{t+1}\right)\right]$ (i.e., it delivers equilibrium prices)

b. $\overline{v}\left(y_t, b_{t+1}, q^\sigma\left(h_m^t, \zeta_t\right)\right) \geq V^\sigma\left(h_m^t, \zeta_t\right)$. This occurs because $\overline{v}$ is the maximum possible continuation value given the price realization $q = q^\sigma\left(h_m^t, \zeta_t\right)$.

[29] In the Online Appendix Section C we define the equilibrium value correspondence and show how it can be computed. To make this proof self contained, we repeat the definition here:

$$\mathcal{E}\left(y_-, b\right) =: \left\{ \left(v, q_-\right) \in \mathbb{R}_2 : \exists \sigma \in \Sigma^*\left(y_-, b\right) : \begin{bmatrix} v = \mathbb{E}\left\{\sum_{t=0}^\infty u\left(c^{\sigma_g}\left(h^t\right)\right)\right\} \\ c_t = y_t - b_t + q_t^{\sigma_m} b_{t+1}^{\sigma_g} \\ b_0 = b \\ q_- = \frac{\mathbb{E}_{y|y_-}\left(1 - d_0^{\sigma_g}(y)\right)}{1+r} \end{bmatrix} \right\}.$$

This set has the utility values for the government and prices for the investors that can be obtained in a subgame perfect equilibrium, given an initial seed value y_-, and initial bonds b. Note that in the model of sovereign debt, we know that the set of prices is $[0, \overline{q}(y_-, b)]$ and the set of values is $[V^{aut}(y_-), \overline{\mathbb{V}}(y_-, b)]$.

Step 1.3 The distribution of prices. The price distribution implied by σ can be defined by a measure Q over measurable sets $A \subseteq \mathbb{R}_+$. More precisely:

$$Q(A) \equiv \int_0^1 \mathbf{1}\left\{q^{\sigma}\left(h_m^t, \zeta_t\right) \in A\right\} d\zeta_t = \Pr\left\{\zeta_t : q^{\sigma}\left(h_m^t, \zeta_t\right) \in A\right\}.$$

Note that condition (a) shows that the support of the distribution is over equilibrium prices; i.e. $Supp\,(Q) \subseteq [0, \overline{q}\,(y_t, b_{t+1})]$. *Step 1.4. The necessary condition.* By changing the integration variables in (A.1), using the definitions above, and conditions (a) and (b):

$$\int_0^{\overline{q}(y_t,b_{t+1})} \left[u\left(y_t - b_t + \hat{q} b_{t+1}\right) + \beta \overline{v}\left(y_t, b_{t+1}, \hat{q}\right)\right] dQ\left(\hat{q}\right) \geq \int_0^1 \left[u\left(y_t - b_t + q^{\sigma}\left(h_m^t, \zeta_t\right) b_{t+1}\right) + \beta V^{\sigma}\left(h_m^t, \zeta_t\right)\right] d\zeta_t$$
$$\geq u\left(y_t\right) + \beta \mathbb{E}_{y_{t+1}|y_t} V^d(y_{t+1});$$

which proves the desired result.

Step 2: Sufficiency ($\Longleftarrow$). Suppose that Q is an equilibrium consistent distribution, after an equilibrium history h_m^t with no default so far. Given that condition (3.2) is satisfied, we need to construct an equilibrium strategy where prices are distributed according to Q and generated h_m^t on its path. *Step 2.1. Preliminaries.* Denote by F_Q the associated cumulative probability function for Q. Denote by $\sigma^*\left(y_t, b_{t+1}, q\right)$ the strategy that achieves the continuation value $\overline{v}\left(y_t, b_{t+1}, q\right)$; i.e.:

$$\sigma^*\left(y_t, b_{t+1}, q\right) \in \underset{\sigma \in \Sigma^*(y_t, b_{t+1})}{\operatorname{argmax}} V^{\sigma}\left(h^0\right) \text{ s.t. } q_0^{\sigma} \leq q\,.$$

As we show in the Online Appendix, Section C, the constraint in this problem, $q_0^{\sigma} \leq q$, is binding. *Step 2.2. Constructing the equilibrium strategy.* Because h_m^t is an equilibrium consistent history, we know there exists an equilibrium profile $\hat{\sigma} = (\hat{\sigma}_g, \hat{\sigma}_m)$ such that $h_m^t \in \mathcal{H}(\hat{\sigma})$. For histories h^s successors of histories $h^t = \left(h^t, y_t, d_t, \hat{b}_{t+1}, \zeta_t, \hat{q}_t\right)$ we define the strategy profile σ for the government as:

$$\sigma_g\left(h^s\right) := \begin{cases} \sigma^d\left(h^s\right) & \text{if } d_t = 1 \text{ or } \hat{b}_{t+1} \neq b_{t+1} \text{ or } \hat{q}_t \notin [0, \overline{q}\left(y_t, b_{t+1}\right)] \\ \sigma^*\left(y_t, b_{t+1}, \hat{q}_t\right)\left(h^s\right) & \text{otherwise.} \end{cases} \tag{A.2}$$

For all $h^s \preceq h_m^t$ we define $\sigma_g\left(h^s\right) := \hat{\sigma}_g\left(h^s\right)$. This strategy, σ_g, prescribes the best continuation equilibrium if the government follows $d_t = 0, b_{t+1}$ and the price that it obtain is an equilibrium price. Alternatively, if the government defaults, chooses a debt level that is different than b_{t+1}, or receives a price that is not an equilibrium price, the government will play default forever after (will be in autarky). In addition, the strategy σ_g that we just defined generates the history h_m^t on its path. Likewise, we define the strategy profile for the market. For histories $\left(h_m^t, \zeta_t\right) = \left(h^t, d_t = 0, b_{t+1}, \zeta_t\right)$, let:

$$q^{\sigma_m}\left(h^t, y_t, d_t = 0, b_{t+1}, \zeta_t\right) = F_Q^{-1}\left(\zeta_t\right) \tag{A.3}$$

where $F_Q^{-1}(\zeta) = \inf\left\{q \in \mathbb{R} : F_Q(q) \geq \zeta\right\}$ is its inverse. For $h^s \preceq h_m^t$ we define $\sigma_m\left(h^s\right) := \hat{\sigma}_m\left(h^s\right)$. For any other history, the market will choose a price of zero. *Step 2.3. Checking incentive compatibility.* Now we need to check that $d_t = 0$ and b_{t+1} is incentive compatible for the candidate strategy profile that we just constructed. Before t, incentive compatibility comes from the fact that h_m^t is equilibrium consistent (i.e. $h_m^t \in \mathcal{H}(\sigma)$). At history h_m^t, for the candidate strategy σ it will be optimal not to default (if we follow strategy σ for all successor nodes) if:

$$\int_0^1 \left[u\left(y_t - b_t + F_Q^{-1}(\zeta)\, b_{t+1}\right) + \beta V^{\sigma}(y_t, b_{t+1}, \zeta_t)\right] d\zeta_t \geq u(y_t) + \beta \mathbb{E}_{y_{t+1}|y_t} V^d(y_{t+1}).$$

where $V^{\sigma}(y_t, b_{t+1}, \zeta_t)$ is the continuation payoff of strategy σ after (y_t, b_{t+1}, ζ_t). This condition is equivalent (if and only if) to:

$$\int_0^{\overline{q}(y_t, b_{t+1})} \left[u(y_t - b_t + \hat{q} b_{t+1}) + \beta \overline{v}(y_t, b_{t+1}, \hat{q})\right] dQ(\hat{q}) \geq u(y_t) + \beta \mathbb{E}_{y_{t+1}|y_t} V^d(y_{t+1}). \tag{A.4}$$

where we use the fact that $F_Q^{-1}(\zeta) =_d$ Uniform $[0, 1]$, and by construction $V^{\sigma}(y_t, b_{t+1}, \zeta_t) = \overline{v}(y_t, b_{t+1}, q_t)$. Condition (A.4) is exactly (3.2) and thus satisfied by hypothesis. Therefore, the government does not want to deviate at t. For any other history, because σ^d and $\sigma^*(y_t, b_{t+1}, \hat{q})$ are subgame perfect equilibrium profiles, the government does not want to deviate. Therefore, $\sigma(h^s)$ defined in (A.2) and (A.3) is an SPE profile (since it is a Nash equilibrium at every possible history) that generates h_m^t and Q on its path. □

A.2 Proposition 2

Proof. *Step 1: Determine the upper bound for probability of $q = 0$.* Denote by $\underline{Q}(\hat{q} = 0)$ the largest probability of a price equal to zero across all equilibrium consistent distributions. To construct $\underline{Q}(\hat{q} = 0)$ after history h_m^t, we need to relax the promise-keeping constraint as much as possible. We do this by focusing on probability distributions $\underline{Q}$ that are binary. These distributions place positive probability only on the worst and best equilibrium prices. As a consequence, $1 - \underline{Q}(\hat{q} = 0)$ is the (lowest) probability of the best equilibrium consistent price. The IC constraint (3.2) needs to hold with equality for this distribution. Thus:

$$\underline{Q}(\hat{q} = 0)\left[u(y_t - b_t) + \beta \mathbb{E}_{y_{t+1}|y_y} V^d(y_{t+1})\right] + \left(1 - \underline{Q}(\hat{q} = 0)\right)\left[\overline{V}^{nd}(b_t, y_t, b_{t+1})\right] = V^d(y_t).$$

This implies that:

$$\underline{Q}(\hat{q} = 0) = \frac{\Delta^{nd}(b_t, y_t, b_{t+1})}{\Delta^{nd}(b_t, y_t, b_{t+1}) + u(y_t) - u(y_t - b_t)},$$

where $\Delta^{nd}(\cdot)$ denotes the maximum utility difference between not defaulting and defaulting (under the best equilibrium), given by $\Delta^{nd}(b_t, y_t, b_{t+1}) := \overline{V}^{nd}(b_t, y_t, b_{t+1}) - V^d(y_t)$. Note further that $\underline{Q}(\hat{q} = 0)$ is bounded away from 1 from an ex-ante perspective (i.e. before the sunspot is realized, but after the government decision has been made) as long as $b_t > 0$.

Step 2: Determine the upper bound for $q = \hat{q}$. Let $p = \Pr(\zeta_t : q(\zeta_t) \leq \hat{q})$. With a reasoning that is similar to the one in Step 1, we can conclude that by focusing on equilibria that have support $q(\zeta_t) \in \{\hat{q}, \overline{q}(y_t, b_{t+1})\}$ we relax the IC constraint (3.2) as much possible (i.e. focus on binary distributions). Thus, we consider equilibria that assigns the best continuation equilibria when $q(\zeta_t) > \hat{q}$ (i.e $q(\zeta_t) = \overline{q}(y_t, b_{t+1})$ and $v(\zeta_t) = \overline{\mathbb{V}}(y_t, b_{t+1})$) and assigns $\overline{v}(y_-, b, \hat{q})$ (the greatest continuation utility consistent with $q \leq \hat{q}$) when $q(\zeta_t) \leq \hat{q}$. The latter because $\overline{v}(y_-, b, \hat{q})$ increasing in $\hat{q}$. Therefore for any such distribution (3.2) holds:

$$p\left[u(y_t - b_t + \hat{q} b_{t+1}) + \beta \overline{v}(y_t, b_{t+1}, \hat{q})\right] + (1 - p)\, V^{nd}(b_t, y_t, b_{t+1}) \geq V^d(y_t).$$

The distribution $\underline{Q}(\hat{q}; b_t, y_t, b_{t+1})$ is defined by the equality of the previous condition. That is:

$$\underline{Q}(\hat{q};b_t,y_t,b_{t+1}) = \frac{\Delta^{nd}(b_t,y_t,b_{t+1})}{V^d(y_t) - [u(y_t - b_t + \hat{q}b_{t+1}) + \beta\overline{v}(y_t,b_{t+1},\hat{q})] + \Delta^{nd}(b_t,y_t,b_{t+1})}.$$

Note that distribution $\underline{Q}(\hat{q};b_t,y_t,b_{t+1})$ is less than 1, only when

$$u(y_t - b_t + \hat{q}b_{t+1}) + \beta\overline{v}(y_t,b_{t+1},\hat{q}) > V^d(y_t).$$

And this happens only when $\hat{q} > \underline{q}(b_t,y_t,b_{t+1})$ where the last inequality comes from the (alternative) characterization of $\underline{q}(b_t,y_t,b_{t+1})$. □

A.3 Proposition 3

Proof. We already know that $\max E(b_t,y_t,b_{t+1}) = \overline{q}(y_t,b_{t+1})$ since the Dirac distribution $\overline{Q}$ over $q = \overline{q}(y_t,b_{t+1})$ is equilibrium consistent. In the same way, we also know that the Dirac distribution $\hat{Q}$ that assigns probability 1 to $q = \underline{q}(b_t,y_t,b_{t+1})$ is equilibrium consistent; this distribution corresponds to a case where both investors and the government ignore the realization of the correlating device, and $\underline{q}(\cdot)$ is exactly the only price that satisfies

$$u\left(y_t - b_t + \underline{q}(b_t,y_t,b_{t+1})\ b_{t+1}\right) + \beta\overline{v}\left(y_t,b_{t+1},\underline{q}(b_t,y_t,b_{t+1})\right) = V^d(y_t). \tag{A.5}$$

In the Online Appendix, Section C, we show that $\overline{v}(y_-,b,q)$ is a concave function in q, which together with the fact that u is strictly concave and $b' > 0$ implies that the function

$$H(q) := u(y_t - b_t + qb_{t+1}) + \beta\overline{v}(y_t,b_{t+1},q)$$

is strictly concave in q. For any distribution $Q \in \mathbb{ECD}(b_t,y_t,b_{t+1})$, let $\mathbb{E}_Q(\hat{q}) = \int \hat{q}dQ(\hat{q})$. Jensen's inequality then implies that

$$\begin{aligned} u(y_t - b_t + \mathbb{E}_Q(q)\, b_{t+1}) + \beta\overline{v}(y_t,b_{t+1},\mathbb{E}_Q(q)) &\underbrace{\geq}_{(1)} \int [u(y_t - b_t + \hat{q}b_{t+1}) + \beta\overline{v}(y_t,b_{t+1},\hat{q})]\, dQ(\hat{q}) \\ &\underbrace{\geq}_{(2)} V^d(y_t) \end{aligned}$$

with strict inequality in (1) if Q is not a Dirac distribution. Then, the definition of $\underline{q}(b_t,y_t,b_{t+1})$ implies that for any distribution $Q \in \mathbb{ECD}(b_t,y_t,b_{t+1})$ we have that:

$$\mathbb{E}_Q(q) \geq \underline{q}(b_t,y_t,b_{t+1});$$

therefore, the minimum expected value is exactly $\underline{q}(b_t,y_t,b_{t+1})$, which is achieved uniquely at the Dirac distribution $\hat{Q}$ (because of the strict concavity of $u(\cdot)$). Finally, knowing that E is naturally a convex set, we then obtain

$$\begin{aligned} E(b_t,y_t,b_{t+1}) &= \left[\min_{Q\in\mathcal{Q}(b_t,y_t,b_{t+1})} \int \hat{q}dQ(\hat{q}), \max_{Q\in\mathcal{Q}(b_t,y_t,b_{t+1})} \int \hat{q}dQ(\hat{q})\right] \\ &= \left[\underline{q}(b_t,y_t,b_{t+1}), \overline{q}(b_t,y_t,b_{t+1})\right] \end{aligned}$$

which is what we wanted to show. □

A.4 Proposition 4

Proof. *Step 1: Determine the bounds for General Random Variables.* To show the bounds on the variance, we rely on the fact that for any random variable X with support in $[a, b] \subseteq \mathbb{R}$ and mean $\mathbb{E}(X) = \mu$, it holds that:

$$\mathbb{V}ar(X) \leq \mu(b + a - \mu) - ab.$$

Moreover, this upper bound in the variance is achieved by a binary distribution P_μ over $\{a, b\}$, with $P_\mu(a) = (b-\mu)/(b-a)$, and of course, $P_\mu(b) = 1 - P_\mu(a)$.

Step 2: Are these bounds Equilibrium Consistent? It Depends. Since the price realization must have support on $[0, \overline{q}(y_t, b_{t+1})]$, after history h^t_m, according to Proposition 1, we know that if $Q : \mathbb{E}_Q(q_t) = \mu$ then $\mathbb{V}_Q(q_t) \leq \mu(\overline{q}(y_t, b_{t+1}) - \mu)$; this bound is achieved by distribution Q_μ with $Q_\mu(0) = \frac{\overline{q}-\mu}{\overline{q}}$. However, this particular distribution may not be equilibrium consistent since it may violate the ex-ante IC for no default, condition (3.2),

$$\int_0^{\overline{q}(y_t,b_{t+1})} [u(y_t - b_t + qb_{t+1}) + \beta\overline{v}(y_t, b_{t+1}, q)]\, dQ_\mu(q) \geq V^d(y_t).$$

Whether this constraint is violated or not will depend on the particular value of $\mu \in \left[\underline{q}(b_t, y_t, b_{t+1}), \overline{q}(y_t, b_{t+1})\right]$. We define q^* as $q^* := \underline{Q}(0) \times 0 + (1 - \underline{Q}(0))\,\overline{q}$.

Step 3: Case 1. IC is not binding for the candidate distribution if the mean is high enough. We first show that if $\mathbb{E}_Q(q_t) = \mu \geq q^*$, then any distribution $Q \in \Delta([0, \overline{q}])$ with $\mathbb{E}_Q(q_t) = \mu$ also satisfies 3.2, and hence the maximum variance is achieved precisely at $\mu(\overline{q} - \mu)$. We now show this. We define

$$D(h^t_m, q_t) := u(y_t - b_t + q_t b_{t+1}) + \beta\overline{v}(y_t, b_{t+1}, q_t) - V^d(y_t);$$

as the difference between the best continuation given a price q_t and history h^t_m, and the worst equilibrium. Remember that $q^* = \underline{Q}(0) \times 0 + (1 - \underline{Q}(0))\,\overline{q}$. Using the definition of $\underline{Q}(0)$, it can be shown that

$$\underline{Q}(0) = \frac{\overline{V}^{nd}(b_t, y_t, b_{t+1}) - V^d(y_t)}{\overline{V}^{nd}(b_t, y_t, b_{t+1}) - V^d(y_t) + u(y_t) - u(y_t - b_t)}.$$

Thus, using the definition of $D(h^t_m, q_t)$ at q^*:

$$\begin{aligned}
D(h^t_m, q^*) &= D\left(h^t_m, \underline{Q}(0) \times 0 + (1 - \underline{Q}(0))\,\overline{q}\right) \\
&> \underline{Q}(0)\, D(h^t_m, 0) + (1 - \underline{Q}(0))\, D(h^t_m, \overline{q}) \\
&= \frac{\overline{V}^{nd}(b_t, y_t, b_{t+1}) - V^d(y_t)}{\overline{V}^{nd}(b_t, y_t, b_{t+1}) - V^d(y_t) + u(y_t) - u(y_t - b_t)} [u(y_t - b_t) - u(y_t)] + \\
&\quad \frac{\overline{V}^{nd}(b_t, y_t, b_{t+1}) - V^d(y_t)}{\overline{V}^{nd}(b_t, y_t, b_{t+1}) - V^d(y_t) + u(y_t) - u(y_t - b_t)} [u(y_t) - u(y_t - b_t)] \\
&= 0.
\end{aligned}$$

Therefore, by the concavity of $\overline{v}(y_t, b_{t+1}, q_t)$,

$$\begin{aligned}\int D\left(h_m^t, q_t\right) dQ\left(q_t\right) &\underbrace{\geq}_{D \text{ is concave in } q} D\left(h_m^t, \mu_Q\right) \\ &\underbrace{\geq}_{D \text{ is increasing in } q} D\left(h_m^t, q^*\right) \\ &> 0.\end{aligned}$$

Thus, when $\mu \geq q^*$ then $\overline{\mathbb{V}ar}\left(q_t\right) = \mu\left(\overline{q}(y_t, b_{t+1}) - \mu\right)$.

We also check that $q^* > \underline{q}$. This holds because $D\left(h_m^t, \underline{q}\right) = 0$ and $D\left(h_m^t, q^*\right) > \underline{Q}(0) D\left(h_m^t, 0\right) + \left(1 - \underline{Q}(0)\right) D\left(h_m^t, \overline{q}\right) = 0$, which then implies that $q^* > \underline{q}$ (because D is strictly increasing in q).

Step 4.1: Case 2. Proposal Violates IC for a Low Mean. We also show that if $Q : \mathbb{E}_Q\left(q_t\right) = \mu < q^*$, then the distribution Q_μ defined as $Q_\mu(0) = \frac{\overline{q} - \mu}{\overline{q}}$ and $1 - Q_\mu(0)$ violates the ex ante no default incentive constraint 3.2. This follows because:

$$\begin{aligned}\mathbb{E}_{Q_\mu}\left[D\left(h_m^t, q_t\right)\right] &= \left(1 - \frac{\mu}{\overline{q}}\right) D\left(h_m^t, 0\right) + \frac{\mu}{\overline{q}} D\left(h_m^t, \overline{q}\right) \\ &= D\left(h_m^t, 0\right) + \frac{\mu}{\overline{q}}\left[D\left(h_m^t, \overline{q}\right) - D\left(h_m^t, 0\right)\right] \\ &< D\left(h_m^t, 0\right) + \frac{\left(1 - Q_\mu(0)\right)\overline{q}}{\overline{q}}\left[D\left(h_m^t, \overline{q}\right) - D\left(h_m^t, 0\right)\right] \\ &= D\left(h_m^t, 0\right) - \frac{D\left(h_m^t, 0\right)}{D\left(h_m^t, \overline{q}\right) - D\left(h_m^t, 0\right)}\left[D\left(h_m^t, \overline{q}\right) - D\left(h_m^t, 0\right)\right] \\ &= 0,\end{aligned}$$

where we use that $\mu < q^*$ and the definition of $q^* = \left(1 - \underline{Q}(0)\right)\overline{q}$. Thus:

$$\mathbb{E}_{Q_\mu}\left[D\left(h_m^t, q_t\right)\right] < 0.$$

This implies that the candidate Q_μ is not an equilibrium consistent price distribution when $\mu < q^*$.

Step 4.2: A New Proposal. To show the second result, following Step 1, we know that we need to restrict attention to binary support distributions; because $D\left(h_m^t, q_t\right)$ is concave, it is easy to show that the support that maximizes the variance (for a given expectation $\mu < q^*$) is $\left\{q_\mu, \overline{q}\right\}$ for some q_μ. Since the no default incentive constraint is binding and we also have a given expectation μ, we need to find q_μ and $\Pr\left(q_\mu\right)$ to solve the following system of equations:

$$\begin{cases}\Pr\left(q_\mu\right) q_\mu + \left(1 - \Pr\left(q_\mu\right)\right)\overline{q} = \mu \\ \Pr\left(q_\mu\right) D\left(h, q_\mu\right) + \left(1 - \Pr\left(q_\mu\right)\right) D\left(h, \overline{q}\right) = 0.\end{cases}$$

Next, note that the second constraint (the no-default incentive constraint), given q_μ is the definition of the infimum distribution

$$\underline{Q}\left(q_\mu\right) = D\left(h_m^t, \overline{q}\right) / \left(D\left(h_m^t, \overline{q}\right) - D\left(h_m^t, q_\mu\right)\right)$$

given in Proposition 2. Using this on the first equation, we obtain one equation in the unknown q_μ :

$$\underline{Q}(q_\mu) q_\mu + (1 - \underline{Q}(q_\mu)) \overline{q} = \mu \iff \frac{D(h_m^t, \overline{q}) - D(h_m^t, q_\mu)}{\overline{q} - q_\mu} = \frac{D(h_m^t, \overline{q})}{\overline{q} - \mu}. \tag{A.6}$$

Because $D(h_m^t, q)$ is increasing in q, the solution q_μ of equation A.6 is increasing in μ in the region where $\mu < q^*$. □

A.5 Corollary 2

Proof. *Step 1. First Order Stochastic Dominance.* Define the function

$$U(Q; b_t, y_t, b_{t+1}) := \int \{u(y_t - b_t + \hat{q} b_{t+1}) + \beta \overline{v}(y_t, b_{t+1}, \hat{q})\} \, dQ(q).$$

Note that thus function is strictly increasing in y_t and strictly decreasing in b_t. Furthermore, the set $\mathcal{Q}(b_t, y_t, b_{t+1})$ can be rewritten as:

$$\mathcal{Q}(b_t, y_t, b_{t+1}) = \left\{ Q \in \Delta([0, \overline{q}]) : U(Q; b_t, y_t, b_{t+1}) \geq V^d(y_t) \right\}.$$

The function $H(q) := u(y_t - b_t + q b_{t+1}) + \beta \overline{v}(y_t, b_{t+1}, q)$ is strictly increasing in q. Therefore, if Q'FOSDQ and $Q \in \mathcal{Q}(b_t, y_t, b_{t+1})$ then $\int H(q) \, dQ' \geq \int H(q) \, dQ \geq V^d(y_t)$. *Step 2. Comparative statistics.* This follows from the fact that

$$U(Q; b_t, y_t, b_{t+1}) - V^d(y_t)$$

is monotonic on y_t (when income is i.i.d.) and on b_t. *Step 3.* $\underline{Q} \notin \mathbb{ECD}(b_t, y_t, b_{t+1})$. Finally we show that $\underline{Q}$ is not an equilibrium consistent distribution. By definition, equation 3.4 cannot be an equilibrium consistent price; this implies that the Lebesgue-Stjeljes measure associated with $\underline{Q}(\cdot)$ has the property that Supp $(Q) = \left[0, \underline{q}(b_t, y_t, b_{t+1})\right]$ and $\underline{Q}(q = 0) = p_0 > 0$, which implies that

$$\begin{aligned}
\int_0^{\overline{q}(y_t, b_{t+1})} \{u(y_t - b_t + \hat{q} b_{t+1}) + \beta \overline{v}(y_t, b_{t+1}, \hat{q})\} \, d\underline{Q}(\hat{q}) &< u\left(y_t - b_t + \underline{q}(\cdot) b_{t+1}\right) + \beta \overline{v}\left(y_t, b_{t+1}, \underline{q}(\cdot)\right) \\
&= V^d(y_t)
\end{aligned}$$

where the last equation comes from the definition of $\underline{q}(\cdot)$ and the function $H(\hat{q})$ is strictly increasing in $\hat{q}$. □

A.6 Main Results of the General Model

The proof Proposition follows the proof of Proposition 3 directly. As in the first section of Online Appendix C Proposition 12, we construct the equilibrium value correspondence as the largest fixed point of the "generating values correspondence" such that for every value correspondence $\mathcal{W}(y_-, b) \subseteq \mathbb{R}^{k+1}$ it gives a set of generating equilibrium values $B(\mathcal{W})(y_-, b) \subseteq \mathbb{R}^{k+1}$. We need the following assumptions: continuity of the utility function and continuity, compact-valuedness and non-emptiness of the feasibility correspondence to guarantee that $\mathcal{E}(y_-, q)$ is non-empty and compact-valued, which implies that $\overline{v}(y_-, b, q)$ and $\underline{U}(b, y)$ are well-defined objects. In addition, we need to assume that the equilibrium value correspondence must is convex-valued. This condition would be enough to guarantee that $\overline{v}(y_-, b, q)$ is concave in q. However, since q enters non-linearly in the contemporaneous utility function of the long-lived player, the convexity of the equilibrium value set is not enough to guarantee that $\mathcal{E} = \mathcal{E}^s$; for this to occur, the contemporaneous utility function $u(\cdot)$ must be concave in q, as is the example of the model of sovereign debt. Armed with these two conditions (the convexity of $\mathcal{E}$ and the concavity of $u(\cdot)$) we can show the result of Proposition 6, which relies on the fact that $\mathcal{E} = \mathcal{E}^s$ plus the concavity of the auxiliary function $D = u + \beta\overline{v}$ to obtain the same results as those in Proposition 3. Using this proposition, $\mathcal{E} = \mathcal{E}^s$ and so are the best continuation function $\overline{v}^s = \overline{v}$ and $\underline{U}^s = \underline{U}$. We change the variable in the integration since ζ enters only through $q(y, \zeta)$. This implicitly defines a measure across prices, according to

$$\int_{\hat{q} \in \mathcal{Q}(y_t, b_{t+1})} \left[u(b_t, y_t, d_t, b_{t+1}, \hat{q}) + \beta\overline{v}(y_t, b_{t+1}, \hat{q}) \right] dQ_t(\hat{q}) \geq \underline{U}(y_t, b_t)$$

which shows the desired result.

Online Appendix to "Robust Predictions in Dynamic Policy Games"

Juan Passadore and Juan Xandri

A Two period example, extended

A.1 Setup and Definitions

In this subsection we formalize some of the results in the game studied in the Introduction, with Extensive form given by Figure 1. In this game, the government decides whether to obtain a sure payoff of $\underline{u}$ if she decides to play "Default". If she decides "Repay", the two agents play the simultaneous move game G, after the realization of a sunspot $\zeta \sim U[0,1]$ (which is payoff irrelevant) . The government plays "columns" and the investor plays "rows". Their action spaces are $\{b_l, b_h\}$ and $\{q_l, q_h\}$ respectively. Payoffs are denoted by (u, v) for both agents respectively. We assume that $\left(u_j, v_j\right) > 0$ for $j \in \{l, h\}$ and $\underline{u} \in (u_l, u_h)$. These assumptions guarantee that the simultaneous move game has two Nash equilibria in pure strategies: (b_l, q_l) and (b_h, q_h).

Strategies, Equilibrium, Outcome. *Strategies* are $\sigma_G = (x, s_1(\cdot))$ for the government and $\sigma_I = s_I(\cdot)$ for the investor, where $x \in \{\text{Repay}, \text{Default}\}$ and $s_G : [0,1] \to \{b_l, b_h\}$ and $s_I : [0,1] \to \{q_l, q_h\}$ are actions played by $i \in \{G, I\}$ in the subgame where $x = \text{Repay}$. $s_i(\cdot)$ is a function of the realization of the sunspot into one of the two actions for each player. A strategy profile $\sigma = (\sigma_G, \sigma_I) = [(x, s_G(\cdot)), s_I(\cdot)]$ is a *SPE* of the game if (a) $x = \text{Repay}$ is optimal for agent 1 given (s_G, s_I) and (b) $(s_G(\zeta), s_I(\zeta))$ is a Nash equilibrium of G for all $\zeta \in [0,1]$. All the SPE $\sigma = [(x, s_G(\cdot)), s_I(\cdot)]$ of this game meet the following conditions: first, for all ζ,

$$(s_G(\zeta), s_I(\zeta)) \in \{(b_l, q_l), (b_h, q_h)\};$$

second, $x = \text{Repay}$ if and only if $\mathbb{E}^{\sigma}_{\zeta}(u_G) \geq \underline{u}$.[30]

An *equilibrium outcome* x is the path of a SPE profile. In this game, for each σ that is an SPE, is given by

$$o^{\sigma}(\zeta) = \begin{cases} \text{Default} & \text{if } x = \text{Default} \\ (\text{Repay}, (s_1(\zeta), s_2(\zeta))) & \text{if } x = \text{Repay} \end{cases}.$$

That is, if the first player chooses Default, then we only observe Default. Or, if the first player chooses Repay, we observe Repay, and the two players play subgame G after observing ζ and a combination of "bond quantities" b and "prices" q. *Because ζ is random, every SPE $\sigma = (\sigma_G, \sigma_I)$ generates a distribution over the Nash Equilibria of the simultaneous move game.*

[30] For example, in the case without sunspots, there are two subgame perfect equilibria:

$$\{[(\text{Default}, b_l), q_l], [(\text{Repay}, b_h), q_h]\}.$$

Given that our objective is to obtain statistical predictions over outcomes, it useful to summarize any equilibrium outcome by a pair (x, Q) where $x \in \{\text{Repay}, \text{Default}\}$ and (Q_l, Q_h) are the probabilities of occurrence of the two Nash Equilibria of the stage game.

A.2 Consistency and Predictions

Equilibrium Consistency. We now introduce the main concept of the paper. We will say that (x, Q) is *equilibrium consistent* if there exists a SPE σ that generates this outcome (x, Q). This concept will help us to make predictions over outcomes of the game conditional on observed histories. The *advantage* of this approach is that *we can obtain predictions over observables without the need to know the particular strategy that generates the outcomes*. So, predictions are robust.

Consistency and Predictions. Suppose that as outside observers we observe $x = \text{Repay}$. Does this place constraints over Q? Yes. In particular,

$$u_l Q_l + u_h Q_h \geq \underline{u}. \tag{A.1}$$

The constraint comes from checking that the decision of agent 1 is optimal, given the distribution Q. For example, after observing Repay the probability of the low equilibrium cannot be equal to one (recall that $\underline{u} \in (u_l, u_h)$). This is the restriction we obtain in Proposition 1, the main result of the paper. Armed with the restriction in (A.1), we can obtain further restrictions over observable variables or over moments of observable variables.

Moments. Using (A.1), we can obtain bounds on the moments across all equilibria for any function $f(q)$ of observable prices. Lets start with expectations. The maximum expectation across all equilibria solves the following linear programming problem:

$$\overline{\mathbb{E}}(f) := \max_{(Q_l, Q_h) \in [0,1]^2} Q_h f(q_h) + Q_l f(q_l)$$

subject to

$$Q_l + Q_h = 1, Q_h \geq 0, Q_l \geq 0$$

and (A.1). Likewise, define $\underline{\mathbb{E}}$ as the minimum over all distributions. Since the objective is a linear functional and the constraint set is convex, then we know that $\left[\underline{\mathbb{E}}(f), \overline{\mathbb{E}}(f)\right]$ is the set of all expected values of f across all SPE distributions. We can also obtain bound on variances. In Proposition 4we obtain bounds over variances of prices given the expectation. In the current example, this amounts again to solve a linear program

$$\overline{\mathbb{V}}(q, \mathbb{E}q) = \max_{(Q_l, Q_h)} Q_h f(q_h)^2 + Q_l f(q_l)^2 - (\mathbb{E}f(q))^2$$

subject to

$$\begin{aligned} Q_h f(q_h) + Q_l f(q_l) &= \mathbb{E}f(q) \\ u_l Q_l + u_h Q_h &\geq \underline{u} \\ Q_l + Q_h &= 1. \end{aligned}$$

General Characterization of Moments. Following the same logic we can characterize all centered moments. Recall that the moment generating function of a random variable q is given by

$$M_q(t) := \mathbb{E}\left(e^{tq}\right)$$

for $t \in \mathbb{R}$.[31] The maximum n-th non centered moment for any function f of observables is:

$$\overline{\mathbb{E}}(f(q^n)) := \max_{(Q_l,Q_h)\in[0,1]^2} \frac{d^n}{dt^n}\left(M_{f(q)}(t)\right)|_{t=0}$$

subject to

$$\begin{aligned} Q_l + Q_h &= 1 \\ Q_h \geq 0, Q_l &\geq 0 \\ u_l Q_l + u_h Q_h &\geq \underline{u}. \end{aligned}$$

Likewise, the can obtain the minimum non centered moment. Note that this is again a linear program, and thus, computationally simple to solve.

B Multiple Equilibrium in Eaton and Gersovitz (1981)

This appendix studies equilibrium multiplicity in the model proposed in section 2. In particular, we characterize the best and worst equilibrium prices (Propositions 7 and 9); the whole set of equilibria (Proposition 8); we provide sufficient conditions for equilibrium multiplicity (Proposition 10); we provide a numerical example for multiplicity; and finally, we discuss which are the implications for deviations from our stylized setting for equilibrium multiplicity. Our results complement the results in Auclert and Rognlie (2016); their paper shows uniqueness in the Eaton and Gersovitz (1981) when the government can save and savings are valued and extends the result for costs of default and the possibility of re-entry.

Preliminaries. For any history h_m^{t+1} we consider the highest and lowest prices

$$\overline{q}^E(h_m^{t+1}) := \max_{\sigma\in\Sigma^*(h_m^{t+1})} q_m\left(h_m^{t+1}\right)$$

$$\underline{q}^E(h_m^{t+1}) := \min_{\sigma\in\Sigma^*(h_m^{t+1})} q_m\left(h_m^{t+1}\right).$$

where $\Sigma^*(h_m^{t+1})$ is the set of equilibria after history h_m^{t+1}. As it will be clear from this section, the set $\Sigma^*(h_m^{t+1})$ is equal to $\Sigma^*(y_t, b_{t+1})$; i.e, the set is pinned down only by y_t, b_{t+1}. The best and worst equilibria turn out

[31] It holds that the n-th non centered moment of $f(q)$ is equal to:

$$\begin{aligned} \mathbb{E}\left(f(q^n)\right) &= \frac{d^n}{dt^n}\left(\mathbb{E}e^{tf(q)}\right)|_{t=0} \\ &= \frac{d^n}{dt^n}\left(M_q(t)\right)|_{t=0}. \end{aligned}$$

to be Markov equilibria and we find conditions for multiplicity. The worst SPE price is zero, and the best SPE price is the one for the Markov equilibrium that is characterized on sovereign debt, such as Arellano (2008) and Aguiar and Gopinath (2006). Our analysis may be of independent interest, because it describes the conditions under which there are multiple Markov equilibria in a sovereign debt model, similar to the one proposed in Eaton and Gersovitz (1981). The importance of this result is that it opens up the possibility of confidence crises in models as in Eaton and Gersovitz (1981). Thus, confidence crises are not necessarily a special feature of the timing in Calvo (1988) and Cole and Kehoe (2000) but rather robust features in most models of sovereign debt. The lowest price $\underline{q}^E(h_m^{t+1})$ can be attained by using a fixed strategy for all histories h_m^{t+1}. This strategy will deliver the utility level of autarky for the government. Thus, the lowest price is associated with the worst equilibrium, in terms of welfare. Likewise, the highest price $\overline{q}^E(h_m^{t+1})$ is associated with a different fixed strategy for all histories (the maximum is attained by the same σ for all h_m^{t+1}) and delivers the highest equilibrium level of utility for the government. Thus, the highest price is associated with the best equilibrium in terms of welfare.

B.1 Lowest Equilibrium Price and Worst Equilibrium

We start by showing that, after any history h_m^{t+1}, the lowest SPE price is equal to zero. Denote by $\mathbf{B}$ the set of assets for the government. We assume that the government cannot save; i.e. $\mathbf{B} \geq 0$.[32]

Proposition 7. *Under the assumption of $\mathbf{B} \geq 0$, the lowest SPE price is equal to zero*

$$\underline{q}^E(h_m^{t+1}) = \underline{q}(y_t, b_{t+1}) = 0$$

and is associated with a Markov equilibrium that achieves the worst level of welfare.

When the government is confronted with a price of zero for its bonds in the present period and expects to face the same price in all future periods, it is best to default. The government cannot benefit from repaying the debt. The proof is simple. We need to show that defaulting after every history is an SPE. Because the game is continuous at infinity, we need to show that there are no profitable one shot deviations when the government uses this strategy. Note, first, that if the government uses a strategy of always defaulting, it is effectively in autarky. In history h_m^{t+1} with income y_t and debt b_t, the payoff of such a strategy is

$$u(y_t) + \beta \mathbb{E}_{y_{t+1}|y_t} V^d(y_{t+1}).$$

Note also that, a one shot deviation involving repayment today has associated utility of

$$u(y_t - b_t) + \beta \mathbb{E}_{y_{t+1}|y_t} V^d(y_{t+1}).$$

Thus, as long as b_{t+1} is non-negative, a one shot deviation of repayment is not profitable. Therefore, autarky is an SPE with an associated price of debt equal to zero.

B.2 Highest Equilibrium Price and Best Equilibrium

We now characterize the best SPE and show that it is the Markov equilibrium studied by the literature on sovereign debt. To find the worst equilibrium price, it was sufficient to use the definition of equilib-

[32] We say that $\mathbf{B} \geq 0$ if for all $b_i \in \mathbf{B}$, $b_i \geq 0$.

rium and the one shot deviation principle. To find the best equilibrium price it will be necessary to find a characterization of equilibrium prices. Denote by $\overline{\mathbb{V}}(y_t, b_{t+1})$ the highest expected equilibrium payoff if the government enters period $t+1$ with bonds b_{t+1} and income in t was y_t. The next lemma provides a characterization of equilibrium outcomes.

Proposition 8. *$x_{t,m} = (q_{t-1}, d_t(\cdot), b_{t+1}(\cdot))$ is an SPE outcome after history h_m^t if and only if the following conditions hold:*

a. The price is consistent

$$q_{t-1} = \frac{\mathbb{E}_{y_t|y_{t-1}}(1 - d_t(y_t))}{1+r}, \tag{B.1}$$

b. IC of the government

$$(1 - d(y_t))\left[u(y_t - b_t + \bar{q}^E(y_t, b_{t+1})b_{t+1}) + \beta\overline{\mathbb{V}}(y_t, b_{t+1})\right] + d(y_t)V^d(y_t) \geq V^d(y_t). \tag{B.2}$$

The proof is omitted. It is a particular case of the main result for the model without sunspots. The main difference comes from the fact that now we do not require h_m^t to be an equilibrium history, and $x_{t,m}$ to be consistent with it. Condition (B.1) states that the price q_{t-1} needs to be consistent with the default policy $d_t(\cdot)$. Condition (B.2) states that a policy $d_t(\cdot), b_{t+1}(\cdot)$ is implementable in an SPE if it is incentive compatible given that following the policy is rewarded with the best equilibrium and a deviation is punished with the worst equilibrium. The argument in the proof follows Abreu (1988). These two conditions are necessary and sufficient for an outcome to be part of an SPE.[33]

Markov Equilibrium. We now characterize the Markov equilibrium that is usually studied in the literature on sovereign debt. The value of a government that has the option to default is given by

$$\overline{\mathbb{V}}(y_-, b) = \mathbb{E}_{y|y_-}\left[\max\left\{\overline{V}^{nd}(b,y), V^d(y)\right\}\right]. \tag{B.3}$$

This is the expected value of the maximum between not defaulting $\overline{V}^{nd}(b,y)$ and the value of defaulting $V^d(y)$. The value of not defaulting is given by

$$\overline{V}^{nd}(b,y) = \max_{b' \geq 0} u(y - b + \bar{q}(y,b')b') + \beta\overline{\mathbb{V}}(y,b'). \tag{B.4}$$

That is, the government repays the debt and obtains a capital inflow (outflow), and from the budget constraint consumption is given by $y - b + q(y,b')b'$; in the next period, the government has the option to default on b' bonds. The value of defaulting is

$$V^d(y) = u(y) + \beta\mathbb{E}_{y'|y}V^d(y'), \tag{B.5}$$

[33]Note that for any history (even those that are *inconsistent* with equilibria) SPE policies are a function of two states: the debt that the government has to pay at time t (b_t) and the income realization from the previous period y_{t-1}. There are two reasons for this. First, the stock of debt and the realization of income from the previous period, summarize the physical environment. Second, the value of the worst equilibrium depends only on the realized income.

and is just the value of consuming income forever. These value functions define a default set

$$D(b) = \left\{ y \in Y : \overline{V}^{nd}(b, y) < V^d(y) \right\}. \tag{B.6}$$

A Markov Equilibrium (with states b, y) is a set of policy functions

$$(c(y,b), d(y,b), b'(y,b)),$$

a bond price function $q(y, b')$ and a default set $D(b)$ such that $c(y, b)$ satisfies the resource constraint; taking as given $q(y, b')$ the government bond policy maximizes $\overline{V}^{nd}$, and the bond price $q(y, b')$ is consistent with the default set

$$q(y, b') = \frac{1 - \int_{D(b')} dF(y' \mid y)}{1 + r}. \tag{B.7}$$

The next proposition states that the best Markov equilibrium is the best SPE.

Proposition 9. *The best SPE is the best Markov equilibrium (i.e.* $\overline{q}(y_t, b_{t+1}) = \overline{q}^E\left(h_m^{t+1}\right)$*).*

Proof. According to proposition 8, the value of the best equilibrium is the expectation with respect to y_t, given y_{t-1}, of

$$\max_{d_t, b_{t+1}} (1 - d_t) \left[u(y_t - b_t + \overline{q}(y_t, b_{t+1}) b_{t+1}) + \beta \overline{\mathbb{V}}(y_t, b_{t+1}) \right] + d_t V^d(y_t).$$

Note that this is equal to the left hand side of (B.3). The key assumption for ensuring that the best SPE is the best Markov equilibrium is that the government is punished with permanent autarky after a default. □

B.3 Multiplicity

Given that the worst equilibrium is autarky, a sufficient condition for the multiplicity of Markov equilibria is any condition that guarantees that the best Markov equilibria has positive debt capacity, which is a standard situation in quantitative sovereign debt models. In general some debt can be sustained as long as there is enough of a desire to smooth consumption, which will motivate the government to avoid default, at least for small debt levels. The following proposition provides a simple sufficient condition for this to be the case. We define $\mathcal{V}^{nd}(b, y; B, \frac{1}{1+r})$ as the value function when the government faces the risk free interest rate $q = \frac{1}{1+r}$ and there is a borrowing limit B as in a standard Bewley incomplete market model. The government has the option to default. This value is not an upper bound on the possible values of the borrower because default introduces state contingency and might be valuable. Our next proposition, however, establishes conditions under which default does not take place.

Proposition 10. *Suppose that for all* $b \in [0, B]$ *and all* $y \in \mathcal{Y}$

$$\mathcal{V}^{nd}(b, y; B, \frac{1}{1+r}) \geq u(y) + \beta \mathbb{E}_{y'|y} V^d(y'). \tag{B.8}$$

Then multiple Markov equilibria exist.

Proof. If the government is confronted with $q = \frac{1}{1+r}$ for $b \leq B$, then condition (B.8) ensures that it will not want to default after any history, which justifies the risk free rate for $b \leq B$. An SPE can implicitly enforce the borrowing limit $b \leq B$ by triggering autarky and setting $q_t = 0$ if $b_{t+1} > B$ ever occurs. Since the debt issuance policy is optimal given the risk free rate, we have constructed an equilibrium. This proves that there is at least one SPE sustaining strictly positive debt and prices. The best equilibrium dominates this one and is Markov, as shown earlier, so it follows that there exists at least one strictly positive Markov equilibrium. Finally, note that we need to check condition (B.8) only for small values of B. However, the existence result then extends an SPE across the entire $\mathbf{B} = [0, \infty)$.[34] $\square$

Example. Suppose there are two income shocks y_L and y_H that follow a Markov chain (a special case is the i.i.d. case). For this case, λ_i denotes the probability of transitioning from state i to state $j \neq i$. We construct an equilibrium where debt is risk free, and the government goes into debt B, stays there as long as its income is low, repays the debt and remains debt free when income is high. Conditional on not defaulting, this bang bang solution is optimal for small enough B. To investigate whether default is avoided, we must compute the values

$$\begin{aligned} v_{BL} &= u(y_L + (R-1)B) + \beta\left(\lambda_L v_{BH} + (1-\lambda_L) v_{BL}\right) \\ v_{BH} &= u(y_H - RB) + \beta\left(\lambda_H v_{0L} + (1-\lambda_H) v_{0H}\right) \\ v_{0L} &= u(y_L + B) + \beta\left(\lambda_L v_{BH} + (1-\lambda_L) v_{BL}\right) \\ v_{0H} &= u(y_H) + \beta\left(\lambda_H v_{0L} + (1-\lambda_H) v_{0H}\right) \end{aligned}$$

where $R = 1 + r$. We write the solution to this system as a function of B. To guarantee that the government does not default in any state, we need to check that $v_{BL}(B) \geq v^{aut}$, $v_{BH}(B) \geq v^{aut}$, $v_{0L}(B) \geq v_L^{aut}$ and $v_{0H}(B) \geq v_H^{aut}$ (some of these conditions can be shown to be redundant). The following proposition provide a simple parametric assumption in which the sufficient conditions hold.

Proposition 11. *A sufficient condition for $v_{BL} \geq v^{aut}, v_{BH} \geq v^{aut}, v_{0L} \geq v_L^{aut}, v_{0H} \geq v_H^{aut}$ that holds for some $B > 0$ is $v'_{BL}(0) > 0, v'_{BH}(0) > 0$. When $\lambda_H = \lambda_L = 1$ this condition simplifies to $\beta u'(y_L) > R u'(y_H)$.*

Note that the simple condition with $\lambda_H = \lambda_L = 1$ is met when u is sufficiently concave or if β is sufficiently close to 1. These conditions ensure that the value from consumption smoothing is high enough to sustain debt.

Proof. Note that we can rewrite the system of Bellman equations as

$$A.v(B) = u(B)$$

Thus, a condition in primitives is

$$v'(0) = A^{-1} u'(0) \geq 0$$

For the special case where $\lambda = 1$, note that

$$\begin{aligned} v_{BH} &= \frac{1}{1-\beta^2}\left(u(y_H - RB) + \beta u(y_L + B)\right) \\ v_{0L} &= u(y_L + B) + \beta v_{BH} \end{aligned}$$

[34] Indeed, it is useful to consider small B and take the limit, which then requires checking only a local condition. The following example illustrates this condition.

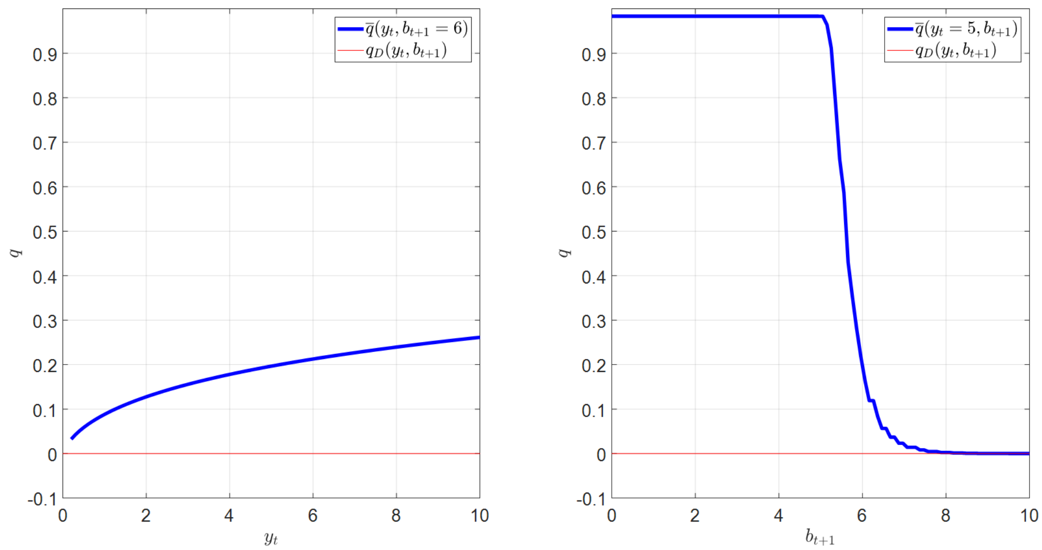

Figure 7: This figure plots the best and worst equilibrium pricing functions: $\overline{q}(y, b')$ and the worst equilibrium price equal to zero.

Then, $v'_{BH}(0) > 0$ implies that $v'_{0L}(0) > 0$. A sufficient condition is $\beta u'(y_L) > Ru'(y_H)$. The intuition is that, the government is credit constrained in the low state, with no debt, and is willing to tradeoff and have lower consumption in the high state. □

A Numerical Illustration. We now numerically illustrate equilibrium multiplicity. The process for log output is given by $\log y_t = \mu + \rho_y \log y_{t-1} + \sigma_y \epsilon_t$ where $\mu = 0.75$. $\sigma_y = 0.3025$, $\rho_y = 0.0945$. The parameters are the same that we use in the main calibration: the discount factor $\beta = 0.953$, CRRA utility with relative risk aversion $\gamma_{RRA} = 2$ and the risk-free interest rate $r = 0.017$. Figure B.3 presents the results. The value functions are the ones in equations (B.3) to (B.7) and the price is given by (B.7). The worst equilibrium has the value of autarky and a price of zero. The best equilibrium is the one studied in quantitative models with short-term debt as in Arellano (2008). Our case is different to Arellano (2008) because there is permanent exclusion after default and there are no direct costs of default. We plot the two price functions, the one of the best equilibrium and the other one is equal to zero (autarky). As it is clear from the right panel, in the best equilibrium for low levels of debt and income debt is risk-free. As we increase the level of debt, the price drops. The price drop is sharp, as is most models with short-term debt.

B.4 Discussion

We close this section with a discussion of conditions under which there is unique or multiple equilibria. First, notice, that sunspots are not needed to generate multiple equilibria. Sunspots may act as a coordinating device to select a particular equilibrium, but we did not use any property of output as a coordinating device to show that either autarky or the best equilibrium are equilibria or that they are different; i.e. we did not use them in any part of Propositions 7, 8, and 10.

Second, as we mentioned before, things are different when the government is allowed to save before default and the punishment is autarky, including exclusion from saving. Under this combination of assumptions, the government might want to repay small amounts of debt to maintain the option to save in the future. As a result, autarky is no longer an equilibrium. Furthermore, a unique subgame perfect equilibrium prevails, as shown by Auclert and Rognlie (2016). Note however, that the government needs to value savings. If savings are not valued, which is a parametric assumption, that means that the value of smoothing consumption with savings is the same as the value of autarky, a condition that is micro-founded in Amador (2013), then autarky will again be an equilibrium; it is easy to modify the proof of Proposition 7 for this case. Furthermore, one can find examples in which savings are not valued and the sufficient conditions for multiplicity of Proposition 10 hold. Finally, note that non-uniqueness holds given a non-equilibrium punishment.

Third, whether or not there are direct costs of default matters for equilibrium multiplicity. For autarky to be an equilibrium, it has to be a dominant strategy to default on any amount of debt that it is allowed to hold if they face a zero price, i.e. to default for all $b \in \mathbf{B}$ if $q = 0$. With default costs, the value of defaulting is lower. Therefore, as with the case of savings, if these costs are large, the government might want to repay small amounts of debt even though the market is offering a zero price of debt in all future period, because the cost of default is too high. Thus, we need to increase the static gain of defaulting for any history. A sufficient condition would then be that $\mathbf{B} > 0$. The lower bound on debt will be increasing in the magnitude of the output costs of default. Stangebye (2019) studies a case that is related to our setting and numerically finds multiple equilibria. The differences in the setup are that in Stangebye (2019) there are output costs of default and, more importantly, there is long term debt, which provides additional forces for equilibrium multiplicity.

C Characterization of $\overline{v}(y_-, b, q_-)$

In this section we characterize the best ex-post continuation value when the income realized is y_- and b bonds are issued at price q_-; i.e.

$$\overline{v}(y_-, b, q_-) := \max_{\sigma \in \Sigma^*(y_-, b)} V(\sigma \mid y_-, b, q_-).$$

The procedure consists of two steps. In the first step, we characterize the set of equilibrium payoffs $\mathcal{E}(y_-, b)$, the values for the government and the prices for the investors. We base our characterization on the concept of self-generation, introduced in Abreu et al. (1990) which has applications for monetary policy Chang (1998b), capital taxation Phelan and Stacchetti (2001) and sovereign lending Atkeson (1991). In the second step, using the set of equilibrium values and prices, $\mathcal{E}(y_-, b)$, we show how to compute $\overline{v}(y_-, b, q)$.

C.1 Step 1: Characterizing the Equilibrium Set $\mathcal{E}(y_-, b)$

We define the equilibrium value correspondence as

$$\mathcal{E}(y_-, b) =: \left\{ (v, q_-) \in \mathbb{R}_2 : \exists \sigma \in \Sigma^*(y_-, b) \; : \; \begin{bmatrix} v = \mathbb{E}\left\{\sum_{t=0}^{\infty} u\left(c^{\sigma_g}\left(h^t\right)\right)\right\} \\ c_t = y_t - b_t + q_t^{\sigma_m} b_{t+1} \\ b_0 = b \\ q_- = \frac{\mathbb{E}_{y|y_-}(1-d_0^{\sigma_g}(y))}{1+r} \end{bmatrix} \right\}.$$

The set $\mathcal{E}(y_-, b)$ has the (utility) values and prices that can be obtained in a SPE, given an initial seed value y_- (recall that income follows a first order Markov process), and the government initially owes b bonds. In period $t = 0$, the government will repay (or not) b by choosing d_0, issuing debt b_1 at a price q_0.To characterize the set of equilibrium payoffs we introduce a procedure that slightly modifies the one first introduced in Abreu et al. (1990).

Step 1.1: Enforceability. Take a bounded, compact-valued correspondence $W : Y \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^2$.

Definition 3. A government *strategy* $(d(\cdot), b'(\cdot))$ *is enforceable* in $W(y_-, b)$ if we can find a pair of functions $v(\cdot,\cdot,\cdot)$ and $q(\cdot,\cdot,\cdot)$ such that:

a. For all $y \in Y, \hat{d}, \hat{b}'$, $\left(v\left(y, \hat{d}, \hat{b}'\right), q\left(y, \hat{d}, \hat{b}'\right)\right) \in W\left(y, \hat{b}'\right)$

b. For all $y \in Y$, the policy $(d(y), b'(y))$ solves the problem:

$$\max_{\hat{d} \in \{0,1\}, \hat{b}' \geq 0} \left(1 - \hat{d}\right) \left\{u\left[y - b + q\left(y, \hat{d}, \hat{b}'\right)\hat{b}'\right] + \beta v\left(y, \hat{d}, \hat{b}'\right)\right\} + \hat{d}\left\{u(y) + \beta \mathbb{E}_{y'|y} V^d(y')\right\}.$$

We refer to the pair $(v(\cdot), q(\cdot))$ as the enforcing values of policy $(d(y), b'(y))$, and we write $(d(\cdot), b'(\cdot)) \in$

$E(W)(y_-,b)$.[35] Further, given the functions $v(\cdot,\cdot,\cdot)$ and $q(\cdot,\cdot,\cdot)$ we define:

$$V^{v(\cdot),q(\cdot)}(b,y) := \max_{\hat{d}\in\{0,1\},\hat{b}'\geq 0} \left(1-\hat{d}\right)\left\{u\left[y-b+q\left(y,\hat{d},\hat{b}'\right)\hat{b}'\right]+\beta v\left(y,\hat{d},\hat{b}'\right)\right\}+\hat{d}\left\{u(y)+\beta\mathbb{E}_{y'|y}V^d(y')\right\}.$$

Definition 4. Given a correspondence $W: Y\times\mathbb{R}_+ \rightrightarrows \mathbb{R}^2$, we define the *generating correspondence* $B(W)$: $Y\times\mathbb{R}_+ \rightrightarrows \mathbb{R}^2$ as:

$$B(W)\left(y,b'\right)=\left\{(v,q)\in\mathbb{R}^2 : \exists\,(d(\cdot),b'(\cdot))\in E(W)\left(y,b'\right) : \left[\begin{array}{c} v=\mathbb{E}_{y'|y}\left[V^{v(\cdot),q(\cdot)}\left(b',y'\right)\right] \\ q=\frac{\mathbb{E}_{y'|y}[1-d(y)]}{1+r}\end{array}\right]\right\}.$$

The idea of $B(W)(y,b')$ is that this is the set of enforceable payoffs given the correspondence $W(\cdot,\cdot)$.

Definition 5. A correspondence $W(\cdot)$ is *self-generating* if for all $y_-\in Y$, $b\geq 0$ it holds that $W(y_-,b)\subseteq B(W)(y_-,b)$.

Step 1.2: A self generating correspondence is an equilibrium correspondence. In this step, we show that if a correspondence of values is self-generating then it belongs to the set of equilibrium values. The proof follows Abreu et al. (1990) and is constructive; to make the manuscript as self contained as possible, we provide a brief discussion of the argument. This is now a standard argument that can be found, for the case without state variables, in different textbooks; for example Mailath and Samuelson (2006). We go back to using the notation $W(y_-,b)$ instead of $W(y,b')$.

Proposition 12. *Any bounded, self-generating correspondence gives equilibrium values: i.e. if* $W(y_-,b)\subseteq B(W)(y_-,b)$ *for all* $y_-\in Y, b\geq 0$ *then* $W(y_-,b)\subseteq\mathcal{E}(y_-,b)$.

Proof. Fix (y_{-1},b_0). Take any pair $(v_{-1},q_{-1})\in W(y_{-1},b_0)$. We would like to show that $(v_{-1},q_{-1})\in\mathcal{E}(y_{-1},b_0)$. To do this, we need to construct an SPE strategy profile $\sigma\in\Sigma^*(y_{-1},b_0)$ that achieves the payoff v_{-1} and in the first period generates the prices q_{-1}.[36] Next, we do just that. Since $W(y_{-1},b_0)\subseteq B(W)(y_{-1},b_0)$, i.e. if W is self generating, then we know we can find functions $(d_0(y_0),b_1(y_0))$, and the values $\left(v_0\left(y_0,\hat{d}_0,\hat{b}_1\right),q_0\left(y_0,\hat{d}_0,\hat{b}_1\right)\right)\in W\left(y_0,\hat{b}_1\right)$ for any $y_0\in Y,\hat{b}_1\geq 0$ such that:

$$(d_0(y_0),b_1(y_0))\in \text{argmax}_{\hat{d}\in\{0,1\},\hat{b}'\geq 0}\left(1-\hat{d}\right)\left\{\left[u\left(y_0-b_0+q_0\left(y,\hat{d},\hat{b}'\right)\hat{b}'\right)+\beta v\left(y,\hat{d},\hat{b}'\right)\right]\right.$$
$$\left.+\hat{d}\left[u(y_0)+\beta\mathbb{E}_{y_1|y_0}V^d(y_0)\right]\right\}$$

[35] We will (sometimes) drop the dependence on d and we will bear in mind that after default the government is not in the market. We will also interchangeably use the notation $W(y_-,b)$ and $W(y,b')$, depending on when which one is most convenient. We find that the notation $W(y,b')$ is most convenient for enforceability, and the notation $W(y_-,b)$ is most convenient for the set of equilibrium payoffs. Furthermore, sometimes we will not include the dependence on $d(\cdot)$ of the set of equilibrium payoffs. Recall that after a default, there is forever in autarky.

[36] Note that v_{-1} is the expected payoff generated by policies $\{d_t,b_{t+1}\}_{t=0}^{\infty}$ given initial bonds b_0 and the seed value for the realization of income y_{-1}; it is the ex-ante payoff from $t=0$ on-wards. In addition, q_{-1} is the price generated by the policy d_0.

i.e., $(d_0(y_0), b_1(y_0))$ is in the argmax of $V^{v_0(\cdot),q_0(\cdot)}(b_0, y_0)$, and the promise keeping constraints

$$v_{-1} = \mathbb{E}_{y_0|y_{-1}}\left\{V^{v_0(\cdot),q_0(\cdot)}(y_0, b_0)\right\},$$

$$q_{-1} = \frac{\mathbb{E}_{y_0|y_{-1}}\left[1 - d_0(y)\right]}{1+r}$$

hold. We define

$$\sigma_g(y_{-1}, b_0) := (d_0(y_0), b_1(y_0))$$

where, for further reference, $h^0 = (y_{-1}, b_0, q_{-1})$,

$$\sigma_m\left(y_{-1}, b_0, y_0, \hat{d}_0, \hat{b}_1\right) = q_0\left(y_0, \hat{d}_0, \hat{b}_1\right).$$

where for further reference $h_m^0 := (y_{-1}, b_0, y_0, d_0, b_1)$. Because:

$$\left(v_0\left(y_0, \hat{d}_0, \hat{b}_1\right), q_0\left(y_0, \hat{d}_0, \hat{b}_1\right)\right) \in W\left(y_0, \hat{b}_1\right),$$

and W is self-generating, we know that for any realization of y_0, we can find policy functions $(d_1(y_1), b_2(y_1))$ and functions $v_1(\cdot,\cdot,\cdot), q_1(\cdot,\cdot,\cdot)$ such that

$$\left(v_1\left(y_1, \hat{d}_1, \hat{b}_2\right), q_1\left(y_1, \hat{d}_1, \hat{b}_2\right)\right) \in B(W)\left(y_1, \hat{b}_2\right)$$

such that the policies $(d_1(y_1), b_2(y_1))$ are in the argmax of $V^{v_1(\cdot),q_1(\cdot)}(b_1, y_1)$ and, the promise keeping constraints hold,

$$v_0\left(y_0, \hat{d}_0, \hat{b}_1\right) = \mathbb{E}_{y_1|y_0}\left(V^{v_1(\cdot),q_1(\cdot)}\left(\hat{b}_1, y_1\right)\right),$$

$$q_0\left(y_0, \hat{d}_0, \hat{b}_1\right) = \frac{\mathbb{E}\left[1 - d_1(y_1)\right]}{1+r};$$

and we define

$$\sigma_g\left(h^1, y_1\right) := (d_1(y_1), b_2(y_1))$$

$$\sigma_m\left(h_m^1\right) := q_1(y_1, d_1, (y_1), b_2(y_1)).$$

Note that $h^1 = (h^0, y_0, b_1, q_0)$ and $h_m^1 = (h^0, y_0, b_1, q_0, y_1, d_1, b_2)$. It is clear that the strategy profiles σ_m and σ_g that are defined for all histories of type h^1 and h_m^1 satisfy the definition of an SPE. By doing this process recursively for all finite t, we can then prove by induction (as in Abreu et al. (1990) original's proof) that this profile is an SPE with initial values (v_{-1}, q_{-1}), as we stated. The finiteness of the value function is guaranteed because the set W is bounded. There are no one shot deviations by construction. □

Proposition 13. *The correspondence $\mathcal{E}(y_-, b)$ is the largest correspondence (in the set order) that is a fixed point of the operator B. That is, $\mathcal{E}(\cdot)$ satisfies:*

$$\mathcal{E}(y_-, b) = B(\mathcal{E})(y_-, b), \tag{C.1}$$

for all $y \in Y$, $b \geq 0$. If another operator $W(\cdot)$ also satisfies condition C.1, then $W(y_-, b) \subseteq \mathcal{E}(y_-, b)$ for all $y \in Y$, $b \geq 0$.

Proof. It is sufficient to show that $\mathcal{E}(y_-, b)$ is self-generating; i.e. $\mathcal{E}(y_-, b) \subseteq B(\mathcal{E})(y_-, b)$. Fix (y_-, b). Take $(v_{-1}, q_{-1}) \in \mathcal{E}(y_-, b)$. We show that $(v_{-1}, q_{-1}) \in B(\mathcal{E})(y_-, b)$. Because $(v_{-1}, q_{-1}) \in \mathcal{E}(y_-, b)$, are equilibrium payoffs, there exists a strategy profile $\sigma = (\sigma_g, \sigma_m)$ associated with the payoffs (v_{-1}, q_{-1}), for a given initial level of income y_- and debt b. Of course, $\sigma = (\sigma_g, \sigma_m) \in \Sigma^*(y_-, b)$. From the definition of a SPE, we know that the policies $d_0(y_0) = d^{\sigma_g}(h^0, y_0)$ and $b'(y_0) = b_1^{\sigma_g}(h^0, y_0)$ are implementable with the following functions $q_0(\cdot,\cdot,\cdot), v_0(\cdot,\cdot,\cdot)$,

$$q_0\left(y_0, \hat{d}_0, \hat{b}_1\right) := q_m^{\sigma}\left(y_0, \hat{d}_0, \hat{b}_1\right)$$

and

$$v_0\left(y_0, \hat{d}_0, \hat{b}_1\right) := V\left(\sigma \mid h^1\left(y_0, \hat{d}_0, \hat{b}_1\right)\right),$$

where

$$h^1\left(y_0, \hat{d}_0, \hat{b}_1\right) := (h^0, y_0, \hat{d}_0, \hat{b}_1, q(y_0, \hat{d}_0, \hat{b}_1)).$$

Moreover, because σ is an SPE strategy profile, it is also an SPE for the continuation game starting with an income realization of y_0 and initial bonds $\hat{b}_1$. Therefore,

$$\left(v_0\left(y_0, \hat{d}_0, \hat{b}_1\right), q_0\left(y_0, \hat{d}_0, \hat{b}_1\right)\right) \in \mathcal{E}\left(y_0, \hat{b}_1\right).$$

Furthermore, the functions $q_0(\cdot,\cdot,\cdot), v_0(\cdot,\cdot,\cdot)$, and the strategies $(d_0(y_0), b'(y_0))$ achieve the payoffs (v_{-1}, q_{-1}). Thus, $(v_{-1}, q_{-1}) \in B(\mathcal{E})(y_-, b)$. Hence $\mathcal{E}(\cdot)$ is a self-generating correspondence, as we wanted to show. □

Step 1.4: Bang Bang Property. In section B of the Online Appendix we characterized the best and worst equilibrium payoffs (prices and utility for the government). These payoffs are the boundaries of $\mathcal{E}(y_-, b)$. We now show that if a policy can be enforced, then it can be enforced with the best and worst continuation payoffs.[37]

Proposition 14. *Suppose that $(d(\cdot), b'(\cdot))$ is an enforceable policy on $\mathcal{E}(y_-, b)$. This policy can be enforced by the following continuation value functions:*

$$v^{BB}\left(y, \hat{d}, \hat{b}'\right) = \begin{cases} \overline{\mathbb{V}}(y, b'(y)) & \text{if } \hat{d} = d(y) = 0 \text{ and } \hat{b}' = b'(y) \\ \mathbb{E}_{y'|y} V^{aut}(y') & \text{otherwise} \end{cases} \tag{C.2}$$

and

$$q^{BB}\left(y, \hat{d}, \hat{b}'\right) = \begin{cases} \overline{q}(y, b'(y)) & \hat{d} = d(y) = 0 \text{ and } \hat{b}' = b'(y) \\ 0 & \text{otherwise.} \end{cases} \tag{C.3}$$

Proof. Note that the functions $v(\cdot), q(\cdot)$ satisfy the restriction

$$\left(v^{BB}\left(y, \hat{d}, \hat{b}'\right), q^{BB}\left(y, \hat{d}, \hat{b}'\right)\right) \in \mathcal{E}\left(y, \hat{d}, \hat{b}'\right)$$

for all $y \in Y$. Since $(d(\cdot), b'(\cdot))$ are enforceable, there exist functions $(\tilde{v}(\cdot), \tilde{q}(\cdot))$ such that for all $y \in Y$

[37] Note that the (singleton) set $\{(v,q)\} = \left\{\left(0, \mathbb{E}_{y|y_-} V^{aut}(y)\right)\right\}$, corresponding to the price and utility of autarky subgame perfect equilibria, is itself self-generating and hence an equilibrium value. Also note that for a given (y_-, b), the values $\{(v,q)\} = \left\{\left(\overline{q}(y_-, b), \overline{\mathbb{V}}(y_-, b)\right)\right\}$ are the expected utility, and the debt price associated with the best equilibrium is also self-generating and hence an equilibrium value.

where $d(y) = 0$ it holds that:

$$u\left(y - b + \tilde{q}\left(y, d(y), b'(y)\right) b'(y)\right) + \beta \tilde{v}\left(y, d(y), b'(y)\right) \geq u\left(y - b + \tilde{q}\left(y, \hat{d}, \hat{b}'\right) \hat{b}'\right) + \beta \tilde{v}\left(y, \hat{d}, \hat{b}'\right) \tag{C.4}$$

for all $y \in Y$ and any alternative policy $\left(\hat{d}, \hat{b}'\right)$. The left hand side of (C.4) is an equilibrium value. Thus, its value must be less than the best equilibrium value for the government, characterized by $q = \overline{q}\left(y, b'(y)\right)$ and $v = \overline{\mathbb{V}}\left(y, b'(y)\right)$. Note that $\overline{\mathbb{V}}\left(y, b'(y)\right)$ denotes the best equilibrium from tomorrow on starting at a debt value of $\hat{b} = b'(y)$ and for an income realization y. From these observations we know that:

$$\begin{aligned} &u\left(y - b + \overline{q}\left(y, b'(y)\right) b'(y)\right) + \beta \overline{\mathbb{V}}\left(y, b'(y)\right) \geq \\ &u\left(y - b + \tilde{q}\left(y, d(y), b'(y)\right) b'(y)\right) + \beta \tilde{v}\left(y, d(y), b'(y)\right). \end{aligned} \tag{C.5}$$

For later reference, recall that

$$\overline{V}^{nd}\left(b, y, b'(y)\right) = u\left(y - b + \overline{q}\left(y, b'(y)\right) b'(y)\right) + \beta \overline{\mathbb{V}}\left(y, b'(y)\right).$$

On the other hand, we know that autarky is the worst equilibrium value (since it coincides with the min-max payoff). Because $\tilde{q}\left(y, \hat{d}, \hat{b}'\right)$ and $\tilde{v}\left(y, \hat{d}, \hat{b}'\right)$ are equilibrium values, it must be the case that:

$$u\left(y - b + \tilde{q}\left(y, \hat{d}, \hat{b}'\right) \hat{b}'\right) + \beta \tilde{v}\left(y, \hat{d}, \hat{b}'\right) \geq u(y) + \beta \mathbb{E}_{y'|y} V^{aut}(y') \tag{C.6}$$

for all $y \in Y$. Combining (C.5) and (C.6) we obtain:

$$u\left(y - b + \overline{q}\left(y, b'(y)\right) b'(y)\right) + \beta \overline{\mathbb{V}}\left(y, b'(y)\right) \geq u(y) + \beta \mathbb{E}_{y'|y} V^{aut}(y') \tag{C.7}$$

which is the enforceability constraint (conditional on not defaulting) of the proposed functions $\left(v^{BB}, q^{BB}\right)$ in equations (C.2) and (C.3). To finish the proof, we need to show that if it is indeed optimal to choose $d(y) = 0$ under the functions $\left(\tilde{v}(\cdot), \tilde{q}(\cdot)\right)$, then it will also be so under functions $\left(v^{BB}(\cdot), q^{BB}(\cdot)\right)$. This is readily given by condition (C.7) since punishment for defaulting coincides with the value of deviating from the bond issue rule $\hat{b} = b'(y)$. Hence, $\left(v^{BB}(\cdot), q^{BB}(\cdot)\right)$ also enforces $\left(d(\cdot), b'(\cdot)\right)$. □

Step 1.5: Monotonicity and an Iterative Procedure. One can show that $W(y_-, b) \subseteq W'(y_-, b)$ implies that $B(W)(y_-, b) \subseteq B(W')(y_-, b)$. This suggests an iterative procedure that can be used to compute the correspondence of equilibrium payoffs, and was first proposed by Abreu et al. (1990) and extended for public state variables in Atkeson (1991), Chang (1998a) and Phelan and Stacchetti (2001). In particular, starting from a compact $W_0(y_-, b)$ and defining $W_n(y_-, b) = B(W_{n-1})(y_-, b)$, it holds that:

$$\mathcal{E}(y_-, b) = \lim_{n \to \infty} W_n(y_-, b).$$

Remark 3. The previous proposition greatly simplifies the characterization of the implementable policies. One can show the following statement, as a simple corollary. A policy $\left(d(\cdot), b'(\cdot)\right)$ is enforceable on $\mathcal{E}\left(y, b'(y)\right)$ if and only if $d(y) = 0$ implies

$$\overline{V}^{nd}\left(b, y, b'(y)\right) \geq V^{d}(y).$$

Remark 4. Note that because we already characterized the best and worst equilibrium values, in Section B, there is no need to perform this iterative procedure, described in *Step 1.5*, for the model of sovereign debt. When the best and worst equilibria are not readily available (for example, in the general model in Section 4 of this paper), the iterative procedure, developed by Judd et al. (2003), would need to be implemented.

C.2 Step 2: Computing $\overline{v}\left(y_{-}, b, q_{-}\right)$

The function $\overline{v}\left(y_{-}, b, q_{-}\right)$ yields the highest expected utility that a government can obtain if given a realization of income y_{-}, they issued b bonds, and the bonds were issued at a price q_{-}. This is the Pareto frontier in the correspondence of equilibrium values. We now discuss how we compute $\overline{v}\left(y_{-}, b, q_{-}\right)$, which can be redefined using the equilibrium value correspondence:

$$\overline{v}\left(y_{-}, b, q_{-}\right) := \max\left\{v : \exists \hat{q} \geq 0 \text{ such that } (v, \hat{q}) \in \mathcal{E}\left(y_{-}, b\right) \text{ and } \hat{q} \leq q_{-}\right\}. \tag{C.8}$$

Note that we focus on a relaxed version of the problem, where we replace the equality $\hat{q} = q$ by the inequality $\hat{q} \leq q$. Proposition 15 enables us to rewrite (C.8) as a linear program. Proposition 16 enables us to compute $\overline{v}\left(y_{-}, b, q_{-}\right)$.

Proposition 15. *For all $q \in [0, \overline{q}(y_{-}, b)]$ the maximum continuation value $\overline{v}\left(y_{-}, b, q_{-}\right)$ solves*

$$\overline{v}\left(y_{-}, b, q_{-}\right) = \max_{d(\cdot)\in\{0,1\}^{Y}} \mathbb{E}_{y|y_{-}}\left[d\left(y\right) V^{d}\left(y\right) + \left[1 - d\left(y\right)\right] \overline{V}^{nd}\left(b, y\right)\right]$$

subject to

$$q_{-} = \frac{\mathbb{E}_{y|y_{-}}\left[1 - d(y)\right]}{1+r}. \tag{C.9}$$

Furthermore, $\overline{v}\left(y_{-}, b, q_{-}\right)$ is non-decreasing and concave in q_{-}.

Proof. *Step 1.1. Programming problem for an arbitrary $\tilde{v}$.* Take any $\tilde{v}$ such that:

$$\tilde{v} \in \left\{v : \exists \hat{q} \geq 0 \text{ such that } (v, \hat{q}) \in \mathcal{E}\left(y_{-}, b\right) \text{ and } \hat{q} \leq q_{-}\right\}.$$

Because $\tilde{v}$ is an equilibrium value, there exists an enforceable policy $\left(\tilde{d}\left(\cdot\right), \tilde{b}\left(\cdot\right)\right)$ and functions $\left(\tilde{v}(\cdot), \tilde{q}(\cdot)\right)$ such that:

$$\begin{aligned}
\tilde{v} &= \mathbb{E}_{y|y_{-}}\left[\left(1 - \tilde{d}\left(y\right)\right)\left[u(y - b + \tilde{q}(y, b'(y))b'(y)) + \beta\tilde{v}(y)\right] + \tilde{d}(y)V^{d}(y)\right] \\
\left(\tilde{d}\left(y\right), \tilde{b}\left(y\right)\right) &\in \arg\max_{d(y), b'(y)} \left(1 - d\left(y\right)\right)\left[u(y - b + \tilde{q}(y, b'(y))b'(y)) + \beta\tilde{v}(y)\right] + d(y)V^{d}(y) \\
\frac{\mathbb{E}_{y|y_{-}}\left[1 - \tilde{d}(y)\right]}{1+r} &\leq q_{-}.
\end{aligned} \tag{C.10}$$

Note that we dropped the dependence of $(\tilde{q}, \tilde{v})$ on the off-path realization of (d, b'), since the punishments from deviation are the worst equilibrium values (Proposition 14). *Step 1.2. Re-writing constraint* (C.10). From Proposition 14, we know that $\left(\tilde{d}\left(y\right), \tilde{b}\left(y\right)\right)$ is also implementable using bang bang continuation values.

Therefore, we can rewrite equation (C.10) as:

$$\left(\tilde{d}\left(y\right),\tilde{b}\left(y\right)\right)\in\arg\max_{\left(d(y),b'(y)\right)}\left(1-d\left(y\right)\right)\left[u(y-b+\bar{q}(y,b'(y))b'(y))+\beta\overline{\mathbb{V}}\left(y,b'\left(y\right)\right)\right]+d(y)V^{d}(y).$$

From Remark 1 we know that[38] for a given choice of $b'(y)$, $(d\left(y\right), b'\left(y\right))$ is enforceable if and only if, the following holds:

$$d(y)=0\implies\overline{V}^{nd}(b,y,b'(y))\geq V^{d}(y).$$

Step 1.3. The programming program for the largest $\tilde{v}$. Therefore, to maximize the arbitrary $\tilde{v}$ the program will now be:

$$\overline{v}\left(y_{-},b,q_{-}\right)=\max_{\left(d(\cdot),b'(\cdot)\right)}\quad\mathbb{E}_{y|y_{-}}\left[\left(1-d\left(y\right)\right)\overline{V}^{nd}\left(b,y,b'\left(y\right)\right)+d(y)V^{d}(y)\right]$$

subject to

$$d(y)=0\implies\overline{V}^{nd}(b,y,b'(y))\geq V^{d}(y)\tag{C.11}$$

$$q_{-}\geq\frac{\mathbb{E}_{y|y_{-}}\left[1-d(y)\right]}{1+r}.$$

Step 1.4. Dropping one constraint. Note that by choosing the optimal $b'(y)$ the constraint (C.11) can be relaxed and we can increase the objective function. Therefore, we can re-write the previous programming problem as:

$$\overline{v}\left(y_{-},b,q_{-}\right)=\max_{d(\cdot)}\quad\mathbb{E}_{y|y_{-}}\left[\left(1-d\left(y\right)\right)\overline{V}^{nd}\left(b,y\right)+d(y)V^{d}(y)\right]$$

subject to

$$d(y)=0\implies\overline{V}^{nd}(b,y)\geq V^{d}(y)\tag{C.12}$$

$$q_{-}\geq\frac{\mathbb{E}_{y|y_{-}}\left[1-d(y)\right]}{1+r}\tag{C.13}$$

Furthermore, note that we can drop constraint (C.12). This because to maximize the function you never want to violate that constraint. *Step 1.5. The price constraint is binding.* Finally, note that if we remove the price constraint, then the agent will choose the default rule to obtain price $\overline{q}\left(y_{-},b\right)$ (the one associated with the best equilibrium). Thus, for any $q<\overline{q}\left(y_{-},b\right)$ this constraint must be binding. *Step 1.6. Increasing in q_{-}.* It is immediate that $\overline{v}\left(y_{-},b,q_{-}\right)$ is weakly increasing in q_{-}. Thus, the programming problem of the government is

$$\overline{v}\left(y_{-},b,q_{-}\right)=\max_{d(\cdot)}\quad\mathbb{E}_{y|y_{-}}\left[\left(1-d\left(y\right)\right)\overline{V}^{nd}\left(b,y\right)+d(y)V^{d}(y)\right]\tag{C.14}$$

subject to

$$q_{-}=\frac{\mathbb{E}_{y|y_{-}}\left[1-d(y)\right]}{1+r}.\tag{C.15}$$

[38] Recall that $\overline{V}^{nd}(b,y,b'(y))$ is defined as

$$\overline{V}^{nd}(b,y,b'(y))=u(y-b+\bar{q}(y,b'(y))b'(y))+\beta\overline{\mathbb{V}}\left(y,b'\left(y\right)\right).$$

Step 2. Concavity. Take $q_0, q_1 \in [0, \overline{q}(y_-, b)]$. We need to show that for every $\lambda \in [0,1]$:

$$\overline{v}(y_-, b, \lambda q_0 + (1-\lambda) q_1) \geq \lambda \overline{v}(y_-, b, q_0) + (1-\lambda)\overline{v}(y_-, b, q_1).$$

Define the functional:

$$G[d(\cdot)] := \mathbb{E}_{y|y_-}\left[d(y) V^d(y) + [1-d(y)] \overline{V}^{nd}(b,y)\right].$$

Let $d_0(y)$ be one of the solutions for the program (C.14) when $q_- = q_0$; likewise, let $d_1(y)$ be one of the solutions of the program when $q_- = q_1$. Define:

$$d_\lambda(y) := \lambda d_0(y) + (1-\lambda) d_1(y).$$

Clearly, this might not be a feasible default policy for the program (C.14); d_λ may belong to $(0,1)$. We solve a relaxed version of the program where $d \in [0,1]$. Note that because the program is linear, the solution is in the boundaries. Note that d_λ is feasible when $q_- = q_\lambda := \lambda q_0 + (1-\lambda) q_1$, since:

$$\begin{aligned}
\frac{\mathbb{E}_{y|y_-}(1-d_\lambda(y))}{1+r} &= \lambda \frac{\mathbb{E}_{y|y_-}(1-d_0(y))}{1+r} + (1-\lambda)\frac{\mathbb{E}_{y|y_-}(1-d_1(y))}{1+r} \\
&= \lambda q_0 + (1-\lambda) q_1 \\
&= q_\lambda.
\end{aligned}$$

Therefore, the optimal continuation value at $q_- = q_\lambda$ must be greater than the objective function evaluated at d_λ. This, because the optimum will be at a corner even in the relaxed problem. This implies that:

$$\begin{aligned}
\overline{v}(y_-, b, q_\lambda) &\geq G[d_\lambda(\cdot)] \\
&= \lambda G[d_0(\cdot)] + (1-\lambda) G[d_1(\cdot)] \\
&= \lambda \overline{v}(y_-, b, q_0) + (1-\lambda)\overline{v}(y_-, b, q_1)
\end{aligned}$$

where we use in the first equality the fact that $G[d(\cdot)]$ is an affine functional in $d(\cdot)$, and in the second one the fact that both $d_0(\cdot)$ and $d_1(\cdot)$ are the optimizers at q_0 and q_1 respectively. □

Proposition 16 solves the programming problem from proposition 15 by reducing it to solving a problem of one equation in one unknown.

Proposition 16. *Given (y_-, b, q_-) there exists a constant $\gamma = \gamma(y_-, b, q_-)$ such that:*

$$\overline{v}(y_-, b, q_-) = \mathbb{E}_{y|y_-}\left[\underline{d}(y) V^d(y) + (1-\underline{d}(y)) \overline{V}^{nd}(b,y)\right]$$

where

$$\underline{d}(y) = 0 \iff \overline{V}^{nd}(b,y) \geq V^d(y) + \gamma(y_-, b, q_-) \text{ for all } y \in Y$$

and γ is the (maximum) solution for the single variable equation:

$$\frac{1}{1+r}\mathbb{P}_{y|y_-}\left\{y : \overline{V}^{nd}(b,y) \geq V^d(y) + \gamma(y_-, b, q_-)\right\} = q_-.$$

Proof. From proposition 15 recall that:

$$\overline{v}\left(y_{-}, b, q_{-}\right)= \quad \max_{d(\cdot)} \quad \mathbb{E}_{y|y_{-}}\left[\left(1-d\left(y\right)\right)\overline{V}^{nd}\left(b, y\right)+d(y)V^{d}(y)\right]$$

subject to

$$q_{-}=\frac{\mathbb{E}_{y|y_{-}}\left[1-d(y)\right]}{1+r}.$$

Note that $d(\cdot)\in\{0,1\}$. We solve a relaxed version of this problem in which $d\left(y\right)\in[0,1]$. Recall that the solution will be in the corners, because we are solving a linear program. The Lagrangian is:

$$\begin{aligned}\mathcal{L}=&\ \mathbb{E}_{y|y_{-}}\left[\left(1-d\left(y\right)\right)\overline{V}^{nd}\left(b,y\right)+d\left(y\right)V^{d}\left(y\right)\right]+\\&+\mathbb{E}_{y|y_{-}}\mu\left(y\right)\left[1-d\left(y\right)\right]\left[\overline{V}^{nd}\left(b,y\right)-V^{d}\left(y\right)\right]\\&+\lambda\left(q_{-}\left(1+r\right)-1+\mathbb{E}_{y|y_{-}}d\left(y\right)\right).\end{aligned}$$

The first order condition with respect to $d\left(y\right)$ is given by:

$$\frac{\partial\mathcal{L}}{\partial\left[d\left(y\right)\right]}=\left[-\overline{V}^{nd}\left(b,y\right)+V^{d}\left(y\right)+\lambda\right]dF\left(y\mid y_{-}\right)$$

where $dF\left(y\mid y_{-}\right)$ denotes the conditional probability of state y. This implies that the optimal default rule is:

$$d\left(y\right)=\begin{cases}0 & \text{if } \overline{V}^{nd}\left(b,y\right)\geq V^{d}\left(y\right)+\lambda\\ 1 & \text{otherwise}\end{cases}$$

for every $y\in Y$, such that $\overline{V}^{nd}\left(b,y\right)\geq V^{d}\left(y\right)$. Defining $\gamma:=\lambda$ we obtain the desired result. We finally need to show that the price constraint is binding at the optimum. This immediate: if this is not the case, then we could increase the objective by defaulting in fewer states of nature. □

D $\overline{v}\left(y_{-}, b, q\right)$ with Restricted Punishments

Here we study the case introduced in Section 4 where equilibrium values must be greater than a function $G\left(y_{-}, b\right)$. In particular, we are interested in finding the equivalent "best equilibrium value" function when restricted to these punishments. The programming problem for $\overline{v}_G$ is given by:

$$\overline{v}_{G}\left(y_{-}, b, q\right)=\max_{(v,q)\in\mathcal{E}\left(y_{-},b\right):v\geq G\left(y_{-},b\right)} v$$

i.e. the best equilibrium value among all equilibrium pairs (v, q) that satisfy the lower bound constraint. In the particular case where $G\left(y_{-}, b\right)=\underline{U}\left(y_{-}, b\right)$ this would correspond to $\overline{v}_G=\overline{v}$. Generalizing the argument in Waki et al. (2018), we can link the problem of finding $\overline{v}_C$ to finding the fixed point of a contraction mapping. Namely, we will study the mapping $T:\mathcal{B}\left(Y\times B\times\mathbb{R}\right)\rightarrow\mathcal{B}\left(Y\times B\times\mathbb{R}\right)$ defined as

$$T\left(f\right)\left(y_{-}, b, q\right)=\sup_{d(\cdot),b'(\cdot),q'(),w(\cdot)}\mathbb{E}_{y}\left[u\left(b, y, d\left(y\right), b'\left(y\right), q\left(y\right)\right)+\beta w\left(y\right)\mid b, y_{-}\right]$$

subject to

$$\begin{cases} u\left(b,y,d\left(y\right),b'\left(y\right),q\left(y\right)\right)+\beta w\left(y\right)\geq G\left(y,b'\left(y\right)\right) & \forall y\ \ (a) \\ \mathbb{E}_y\left[T\left(b_{s+1},y_{s+1},d_{s+1},b_{s+2}\right)+\delta q'\left(y\right)\right]=q\left(y\right) & \forall y\ \ (b) \\ w\left(y\right)\leq f\left(y,b'\left(y\right),q'\left(y\right)\right) & \forall y\ \ (c) \end{cases}$$

where (a) is the incentive constraint, (b) is the moment condition for q and (c) are the bounds for continuation value, that must be above C for all values of y and below the candidate for best equilibrium value $f\left(y_{-},b,q\right)$. It is easy to check that T satisfies Blackwell conditions (monotonicity and discount) and is hence a contraction mapping with modulus β, and hence it has a fixed point $f^{*}\left(y_{-},b,q\right)$. See that in in the sup program of T we will always have $w\left(y\right)=f\left(y,b'\left(y\right),q'\left(y\right)\right)$ and then it is easy to see that the fixed point $f^{*}$is self-generating (see Waki et al. (2018) for an extended argument for this).

E Derivations and Proofs: Section 4.2

E.1 Worst Stationary Equilibrium: Derivation of (4.3)

Proposition 17. *The worst stationary equilibrium payoff is given by*

$$U^{s}\left(y\right)=-\frac{\frac{\chi}{\kappa^{2}}}{2\left(1+\frac{\chi}{\kappa^{2}}\right)}\left(y^{2}+\beta\frac{\sigma_{y}^{2}}{1-\beta}\right).$$

Proof. The proof is as follows. The payoff $\underline{U}^{s}(y)$ and optimal spot strategy solves $\pi(y)$ given π^{e} solves

$$\underline{U}^{s}(y)=\min_{\pi(y)}\frac{1}{2}\left(\pi(y)-y\right)^{2}+\frac{\chi}{2\kappa^{2}}\left(\pi(y)-\beta\pi^{e}\right)^{2}+\beta\mathbb{E}_{y}\left[\underline{U}^{s}(y)\right]. \tag{E.1}$$

From the first order condition for $\pi(y)$ we get that

$$\pi(y)=\frac{1}{1+\frac{\chi}{\kappa^{2}}}y, \tag{E.2}$$

which implies that $\pi^{e}=\mathbb{E}_{y}\left(\frac{1}{1+\frac{\chi}{\kappa^{2}}}y\right)$ is equal to zero. Plugging in (E.2) into (E.1), we get that

$$\underline{U}^{s}(y)=\frac{1}{2}\left(\frac{1}{1+\frac{\chi}{\kappa^{2}}}y-y\right)^{2}+\frac{1}{2}\frac{\chi}{\kappa^{2}}\left(\frac{1}{1+\frac{\chi}{\kappa^{2}}}y\right)^{2}+\beta\mathbb{E}_{y}\left[\underline{U}^{s}(y)\right]$$

$$\mathbb{E}_{y}\left[\underline{U}^{s}(y)\right]=\mathbb{E}_{y}\left[\frac{1}{2}\left(\frac{1}{1+\frac{\chi}{\kappa^{2}}}y-y\right)^{2}+\frac{1}{2}\frac{\chi}{\kappa^{2}}\left(\frac{1}{1+\frac{\chi}{\kappa^{2}}}y\right)^{2}\right]+\beta\mathbb{E}_{y}\left[\underline{U}^{s}(y)\right].$$

From this system we obtain $\underline{U}^{s}(y)$ and $\underline{U}^{s}=\mathbb{E}_{y}\left[\underline{U}^{s}(y)\right].$ □

E.2 Best Equilibrium: Derivation of (4.4)

In this subsection we show that the best continuation equilibrium given expected inflation π^e is given by:

$$\overline{v}(\pi^e) = -\frac{\delta}{2}(\pi^e)^2 - \gamma. \tag{E.3}$$

The proof is as follows. The best equilibrium solves given an expected inflation π^e solves

$$\overline{v}(\pi^e) = \max_{\pi(y),\pi_+^e(y),w(y)} \mathbb{E}_y\left[-\frac{\chi}{2\kappa^2}\left(\pi(y) - \beta\pi_+^e(y)\right)^2 - \frac{1}{2}\left(\pi(y) - y\right)^2 + \beta w(y)\right]$$

subject to

$$-\frac{\chi}{2\kappa^2}\left(\pi(y) - \beta\pi_+^e(y)\right)^2 - \frac{1}{2}\left(\pi(y) - y\right)^2 + \beta w(y) \geq G(y) \tag{E.4}$$

$$\mathbb{E}_y\left[\pi(y)\right] = \pi^e \tag{E.5}$$

$$w(y) \leq \overline{v}(\pi_+^e(y)).$$

First, is easy to note that in the optimum of the right hand side of the Bellman equation above, we will always have $w(y) = \overline{v}(\pi_+^e(y))$, since it relaxes constraint (E.4) and increases the objective. Second, we conjecture that condition (E.4) is in fact, non binding. Intuitively, this is because at the fixed point, in the maximization problem we can readily choose $\pi_+^e(y) = 0$ and choose the myopic best response $\pi(y)$, in the same way we constructed the stationary equilibrium above.Then, we can rewrite this fixed point problem as a relaxed Bellman equation best equilibrium payoff, with only constraint E.5 active:

$$\overline{v}(\pi^e) = \max_{\pi(\cdot),\pi_+^e(y),w(y)} \mathbb{E}_y\left\{-\frac{\chi}{2\kappa^2}\left(\pi(y) - \beta\pi_+^e(y)\right)^2 - \frac{1}{2}\left(\pi(y) - y\right)^2 + \beta\overline{v}(\pi_+^e(y))\right\} \tag{E.6}$$

subject to

$$\mathbb{E}_y\left(\pi(y)\right) = \pi^e.$$

Plugging (E.3) into (E.6), the first order conditions with respect to $(\pi(\cdot), \pi_+^e(y), w(y))$ are given by:

$$\pi(y) - y + \frac{\chi}{\kappa^2}\left(\pi(y) - \beta\pi_+^e(y)\right) = \lambda \tag{E.7}$$

$$-\frac{\chi}{\kappa^2}\beta\left(\pi(y) - \beta\pi_+^e(y)\right) + \beta\delta\pi_+^e(y) = 0 \tag{E.8}$$

$$\mathbb{E}_y\left(\pi(y)\right) = \pi^e.$$

If N is the number of values that y can take, given δ, this is a system of $2N+1$ equations in the unknowns $(\pi(y), \pi_+^e(y), \lambda)$. Solving for $(\pi(\cdot), \pi_+^e(y), w(y))$ and plugging the solution into (E.6) we find that the right hand side of (E.6) is quadratic, and we obtain a system of two equations for δ and γ, which are given by

$$\delta = \frac{1 + \frac{\chi}{\kappa^2}(1-\beta) + \sqrt{1 + 2\frac{\chi}{\kappa^2}(1+\beta) + \left(\frac{\chi}{\kappa^2}\right)^2 (1-\beta)^2}}{2}, \tag{E.9}$$

$$\gamma = \frac{\delta\varphi\frac{\chi}{\kappa^2}}{2(1-\beta)}\sigma_y^2 \tag{E.10}$$

$$\varphi := \frac{1}{\delta\left(1+\frac{\chi}{\kappa^2}\right) + \frac{\chi}{\kappa^2}\beta}. \tag{E.11}$$

E.3 Equilibrium Consistency: Bounds for π^e

Given y_t a pair of (π_t, π^e_{t+1}) is equilibrium consistent if and only if

$$\left(1+\frac{\chi}{\kappa^2}\right)\pi_t^2 - 2\beta\chi\pi_t\pi^e_{t+1} + \beta\left(\beta\frac{\chi}{\kappa^2}+\delta\right)\left(\pi^e_{t+1}\right)^2 - 2y_t\pi_t + \frac{y_t^2}{1+\chi} - \eta \leq 0. \tag{E.12}$$

where δ is given by (E.9). First, note that because the discriminant is negative

$$\begin{aligned}\Delta &= (-2\beta\chi)^2 - 4\left(1+\frac{\chi}{\kappa^2}\right)\beta\left(\beta\frac{\chi}{\kappa^2}+\delta\right)\\ &= -4\frac{\beta}{\psi} < 0,\end{aligned}$$

we know that the pairs (π_t, π^e_{t+1}) for which equation (E.12) holds form an ellipse. Second, we compute the bounds on expected inflation, by solving (E.12), and we obtain two bounds $\left\{\overline{\pi}^e_{t+1}, \underline{\pi}^e_{t+1}\right\}$ as a function of (π_t, y_t), such that

$$\pi^e_{t+1} \in \left[\overline{\pi}^e_{t+1}(\pi_t, y_t), \underline{\pi}^e_{t+1}(\pi_t, y_t)\right].$$

These bounds are the solution to the quadratic equation (E.12) with equality, and are given by

$$\overline{\pi}^e_{t+1}(\pi_t, y_t) = \left(\beta\frac{\chi}{\kappa^2}\delta\varphi\right)\pi_t + \sqrt{-\delta^2\varphi\pi_t^2 + 2\delta\varphi y_t\pi_t - \delta\varphi\left(\frac{y_t^2}{1+\frac{\chi}{\kappa^2}} - \eta\right)}$$

$$\underline{\pi}^e_{t+1}(\pi_t, y_t) = \left(\beta\frac{\chi}{\kappa^2}\delta\varphi\right)\pi_t - \sqrt{-\delta^2\varphi\pi_t^2 + 2\delta\varphi y_t\pi_t - \delta\varphi\left(\frac{y_t^2}{1+\frac{\chi}{\kappa^2}} - \eta\right)}.$$