Online Appendix to "The Macroeconomics of Hedging Income Shares"

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E Financial Contract

E.1 Optimal Contract

In this section we provide the main idea for the optimal contract. Suppose that there is one period. There is a risk neutral principal which can provide insurance to the entrepreneurs. Suppose there are three possible idiosyncratic shocks $g_L < g_M < g_H$, with probability p_i.[27] At the beginning of the period, before knowing the realization of g_i, the firm can enter an insurance contract with the financial intermediary. The firm's profits are $\alpha Y g_i$. In absence of insurance the entrepreneur's utility is:

$$\mathbb{E}_i[u(e_i)] = \mathbb{E}_i[u(\alpha Y g_i)]$$

The principal (financial intermediary) can sign a contract and offer insurance to the entrepreneur.

Full Insurance. In the benchmark when g_i is observable the contract is simple: the principal "buys" all the proceeds of the production with a lump sum payment of J. Then, after the shock is realized the entrepreneur hands over the profits to the principal. Because the principal must break-even, it must be the case that $\mathbb{E}_i(\alpha Y g_i) - J = 0$. Thus, the utility of the entrepreneur in this case with full insurance is:

$$\mathbb{E}_i[u(e_i^O)] = E_i[u(J)] = u(\mathbb{E}_i[\alpha Y g_i]) > \mathbb{E}_i[u(\alpha Y g_i)] = \mathbb{E}_i[u(e_i)]$$

Moral Hazard. However, the entrepreneur is subject to moral hazard problem because g_i *is not observable*. The entrepreneur can report an alternative value of g_i, say $g_{i'}$, and keep the difference for herself. Therefore, any contract adds a constraint so that the entrepreneurs reveals the true realization of g_i (incentive compatibility). But, transforming

[27] The contract with three shocks can be generalized to any finite numbers of shocks. With only 2 shocks the results might not generalize to more states.

these "stolen" profits into consumption is not for free. Each unit of stolen profit transforms into consumption at the rate $0 \leq \psi \leq 1$. Thus, when the entrepreneur steals profits, she obtains an additional consumption of only $\psi \alpha Y(g_i - g_{i'})$. To force truthful revelation the principal must hand over additional payments d_i contingent on the realization of g_i. Since the entrepreneur will not lie in equilibrium, her consumption is $e_i^C = J + d_i$, while because the principal must break even the contract must also satisfy: $\mathbb{E}_i(\alpha Y g_i - d_i) - J = 0$, where we normalize the outside option of the principal to zero without loss of generality. As a result, the optimal contract solves:

$$\max_{\{J,d_i\}} \mathbb{E}_i u(J + d_i)$$

$$\text{st.} \quad \psi \alpha Y(g_i - g_{i'}) + d_{i'} + J \leq J + d_i; \quad \forall i, i'$$

$$\mathbb{E}_i(\alpha Y g_i - d_i) - J = 0.$$

The first set of constraints are the *incentive compatibility*, or truth telling, constraints. Notice that only the adjacent constraints matter. To see this, consider that the entrepreneur would never lie when she observes the low shock. So, only the following can be binding:

$$\psi \alpha Y(g_H - g_M) + d_M \leq d_H$$
$$\psi \alpha Y(g_M - g_L) + d_L \leq d_M$$
$$\psi \alpha Y(g_H - g_L) + d_L \leq d_H$$

Adding the first two inequalities:

$$\begin{aligned} \psi \alpha Y(g_H - g_M) + d_M + \psi \alpha Y(g_M - g_L) + d_L \leq & \quad d_H + d_M \\ \psi \alpha Y(g_H - g_L) + d_L \leq & \quad d_H \end{aligned}$$

Thus, the third constraint is irrelevant. In general this is a version of the single crossing property, and it can be generalized to any arbitrary number of idiosyncratic shocks. Rewriting the problem we have:

$$\max_{\{J,d_i\}} \sum_i p_i u(J + d_i)$$

$$\text{st.} \quad \psi \alpha Y(g_H - g_M) + d_M \leq d_H$$

$$\psi \alpha Y(g_M - g_L) + d_L \leq d_M$$

$$\sum_i p_i(\alpha Y g_i - d_i) - J = 0$$

Let λ be the multiplier in the break even constraint and μ_i the multiplier in each incentive compatibility. Taking first order conditions:

$$\begin{aligned}
\sum_i p_i u'(J + d_i) &= \lambda \\
\gamma_L u'(J + d_L) &= \gamma_L \lambda + \mu_M \\
\gamma_M u'(J + d_M) &= \gamma_M \lambda + \mu_H - \mu_M \\
\gamma_H u'(J + d_H) &= \gamma_H \lambda - \mu_H
\end{aligned}$$

It is clear that $\mu_L = \mu_H = 0$ cannot be a solution because it violates the IC constraints. Now, suppose $\mu_M = 0$, while $\mu_H > 0$. Then it must be that $d_L = d_M$. If $d_L > d_M$, the IC constraint implies

$$\psi \alpha Y(g_M - g_L) + d_L - d_M < 0$$

Which is a contradiction. If $d_L < d_M$ a small increase in d_L accompanied by a small reduction on d_M, keeping the break even constraint satisfied, generates a welfare change of:

$$\gamma_L d_L [u'(e_L) - u'(e_M)] > 0$$

which is true because $u''(.) < 0$ and $e_L < e_M$, thus increasing welfare. A similar argument can be used to show that $\mu_M > 0$ and $\mu_H = 0$ is not possible either. As a result, because μ_M and μ_H are both strictly positive, we must have:

$$\begin{aligned}
\psi \alpha Y(g_H - g_M) &= d_H - d_M \\
\psi \alpha Y(g_M - g_L) &= d_M - d_L
\end{aligned}$$

It is easy to see that $d_i = \psi \alpha Y g_i$, together with $J = (1 - \psi)\mathbb{E}_i(\alpha Y g_i)$, is a solution for all the equations. And since the problem has a unique solution, it must be the solution. This contract can be interpreted as an equity contract. Each entrepreneur sells a share $1 - \psi$ of her firm to the intermediary and uses the proceeds to buy an indexed stock market financial instrument. This completely smooths out a proportion $(1 - \psi)$ of the idiosyncratic risk. However, to prevent stealing not all the shares can be sold, the entrepreneur must retain a proportion ψ of her shares, which is her "*skin in the game*". This is the best insurance possible with only short term contracts. Note that here we assume that there was no aggregate risk. This result would not be affected by it, since it would affect all the IC constraints proportionally. It would only change the pricing of J.

E.2 Constrained Efficiency

In this section we show that the equilibrium in the two period model is constrained efficient. The notion of constrained efficiency follows Geanakoplos and Polemarchakis (1986) and Stiglitz (1982), and provides the planner with the same instruments as the market. In particular, the planner can intervene redistributing consumption across aggregate states with a lump sum transfer $T(s)$. Consumption for the consumer and the entrepeneur are given by:

$$c_2(s) = T(s) + (1 - \alpha(s))Y_2(s)$$
$$e_2(s,i) = -T(s) + \alpha(s)g_i Y_2(s).$$

Without loss of generality, and to follow the notation of the paper, we define

$$T(s) := \frac{\phi(s)}{Y_2(s)}.$$

Planning Program. The planner solves

$$\max_{\{e_1,c_1,\phi(s),c_2(s)\}_{s\in S}} \frac{e_1^{1-\gamma}}{1-\gamma} + \mathbb{E}_{i,s}\frac{e_2(s,i)^{1-\gamma}}{1-\gamma}$$

$$c_1 + e_1 = Y_1 \tag{65}$$
$$c_2(s) + e_2(s) = Y_2(s) \tag{66}$$
$$c_2(s) = \phi(s)Y_2(s) + (1 - \alpha(s))Y_2(s) \tag{67}$$
$$e_2(s,i) = -\phi(s)Y_2(s) + \alpha(s)g_i Y_2(s) \tag{68}$$
$$e_2(s) = \mathbb{E}_i e_2(s,i) \tag{69}$$
$$\frac{c_1^{1-\gamma}}{1-\gamma} + \mathbb{E}_s\frac{c_2(s)^{1-\gamma}}{1-\gamma} \geq \underline{u} \tag{70}$$

for all (s,i) . Equations (65) and (66) are the resource constraints for periods one and two. Equations (67) and (68) pin down consumption for the consumer and the entrepreneur in period two. The last constraint maps the Pareto frontier. Lets re-write the program in terms of consumption of the entrepreneur:

$$\max_{\{e_1,\phi(s)\}_{s\in S}} \frac{e_1^{1-\gamma}}{1-\gamma} + \mathbb{E}_{i,s}\frac{\left(-\phi(s)Y_2(s) + \alpha(s)g_i Y_2(s)\right)^{1-\gamma}}{1-\gamma}$$

$$\frac{\left(Y_1 - e_1\right)^{1-\gamma}}{1-\gamma} + \mathbb{E}_s \frac{\left(\phi(s)Y_2(s) - \alpha(s)Y_2(s)\right)^{1-\gamma}}{1-\gamma} \geq \underline{u}.$$

The first order conditions are

$$\begin{aligned} e_1 : e_1^{-\gamma} - \lambda \left(Y_1 - e_1\right)^{-\gamma} = 0 \\ \phi(s) : -Y_2(s)\Pi(s)\mathbb{E}_i \left(-\phi(s)Y_2(s) + \alpha(s)g_i Y_2(s)\right)^{-\gamma} \\ + \lambda Y_2(s)\Pi(s) \left(\phi(s)Y_2(s) + (1-\alpha(s))Y_2(s)\right)^{-\gamma} = 0 \end{aligned}$$

Thus, for every state the ratio of consumption is equal to:

$$\frac{e_1^{-\gamma}}{c_1^{-\gamma}} = \frac{\mathbb{E}_i \left(-\phi(s)Y_2(s) + \alpha(s)g_i Y_2(s)\right)^{-\gamma}}{\left(\phi(s)Y_2(s) + (1-\alpha(s))Y_2(s)\right)^{-\gamma}}.$$

This is exactly the same relative allocation of consumption. Note that the level of consumption will depend on the reservation utility $\underline{u}$. Thus, for a particular choice of $\underline{u}$ we can recover the allocation of the competitive equilibrium.

Efficiency We summarize the discussion in the following proposition. A competitive equilibrium is Pareto Efficient if there exists some $\underline{u}^*$ such that the allocation of the competitive equilibrium for a given initial distribution of wealth (α_1, ϕ_1)[28] is equal to the solution of the planning problem for a level of reservation utility $\underline{u}^*$.

Proposition 5. *The competitive equilibrium with id risk is the solution of the planning problem when*

$$\underline{u} = \frac{\left(Y_1 - e_1^{IM}\left(\alpha_1, \phi_1\right)\right)^{1-\gamma}}{1-\gamma} + \mathbb{E}_s \frac{\left(\phi^{IM}(s)\left(\alpha_1, \phi_1\right) Y_2(s) - \alpha(s)Y_2(s)\right)^{1-\gamma}}{1-\gamma}. \tag{71}$$

Thus, the competitive equilibrium is constrained efficient.

The proof is immediate. For the level $\underline{u}$ defined in (71): the participation constraint of the consumer holds with equality, and the allocation of incomplete markets meets all of the first order conditions hold. In the case that the planner has more instruments, in particular, it can perfectly control consumption, then the planning problem will be the same one as the complete markets allocation. Thus, we need to make a choice regarding which problem we are focusing on. Note that this result is coming from the fact that in the two period problem we are not micro-founding the amount of id risk that the entrepreneurs

[28] Note that this pins down the initial assets of each one the consumer and the entrepreneur and initial income (that depends on the capital and labor share).

face. This is also the case in the infinite horizon. The source of the inefficiency, and some other papers in the literature is in there.

F Capital Adjustment Costs

Suppose there are capital adjustment costs given by:

$$\frac{\varpi}{2}\left(\frac{k'}{W^e(k,i)} - v\right)^2 W^e(k,i)$$

For some $v > 0$, the entrepreneur solves

$$V^e(E,k;s,i) = \max_{\{e,E(s'),k'\}} \{u(e(s,i)) + \beta \mathbb{E}_{s',i'}[V^e(E(s'|s),k';s',j')|s]\}$$

subject to

$$e(s,i) + k' + \frac{\varpi}{2}\left(\frac{k'}{W^e(k,i)} - v\right)^2 W^e(k,i) + \sum_{s'} p(s'|s)E(s'|s) \leq E(s) + r(s)kg_i$$

The variables and the interpretations are the same as before.

F.1 Entrepreneur Solution

First notice that

$$\frac{\partial(\frac{\varpi}{2}\left(\frac{k'}{W^e(k,i)} - v\right)^2 W^e(k,i))}{\partial W^e(k,i)} = -\frac{\varpi}{2}\left(\left(\frac{k'}{W^e(k,i)}\right)^2 - v^2\right)$$

The foc's imply

$$p(s'|s)u'(e(s,i)) = \beta \Pi(s'|s)\mathbb{E}_i\left[u'(e(s',i))\left(1 + \frac{\varpi}{2}(\left(\frac{k''}{W^e(s')}\right)^2 - v^2)\right)\right]; \quad \forall s,s'$$

$$[1 + \varpi\left(\frac{k'}{W^e} - v\right)]u'(e(s,i)) = \beta \mathbb{E}_{s',i}\left[u'(e(s',i))\left(1 + \frac{\varpi}{2}(\left(\frac{k''}{W^e(s')}\right)^2 - v^2)\right) r(s')g_i\right]$$

Notice that $q = [1 + \varpi \left(\frac{k'}{W^e(k,i)} - v\right)]$, is similar to the typical investment q. As before, guess that the solution is characterized by:

$$\begin{aligned} e(s,i) &= (1-\vartheta(s))W^e(s,i,k) \\ k'(s,i) &= \nu(s)\vartheta(s)W^e(s,i,k) \\ E(s'|s,i) &= \phi^e(s'|s)E_1(s,i) \end{aligned}$$

Total entrepreneur's wealth is as before:

$$W^e(s,i,k) = E(s,i) + r(s)g_i k.$$

The assumed shape for the adjustment cost is useful because we know k'/h^e is independent of i and equal to $v(s) = \nu(s)\vartheta(s)$. Main difference now is that, using budget constraint:

$$E_1(s,i) \equiv [\vartheta(s)(1-\nu(s)) - \frac{\varpi}{2}\left(v(s)-v\right)^2]W^e(s,i,k)$$

And again, we must have $\sum_{s'|s} p(s'|s)\phi^e(s'|s) = 1$. Note that the law of motion of wealth is, as before:

$$\begin{aligned} W^e(s',i',k') &= E(s') + r(s')g_{i'}k' \\ W^e(s',i',k') &= [\vartheta(s)(1-\nu(s)) - \frac{\varpi}{2}\left(v(s)-v\right)^2]\phi^e(s'|s)W^e(s,i,k) + \nu(s)\vartheta(s)r(s')g_{i'}W^e(s,i,k) \\ W^e(s',i',k') &= \vartheta(s)[[(1-\nu(s)) - \frac{\varpi}{2\vartheta(s)}\left(v(s)-v\right)^2]\phi^e(s'|s) + \nu(s)r(s')g_{i'}]W^e(s,i,k) \quad (72) \end{aligned}$$

Using this and putting both Euler equations together, it implies:

$$\frac{\mathbb{E}_{s',i|s}\left[u'(e(s',i))\left(1+\frac{\varpi}{2}(v(s')^2-v^2)\right)r(s')g_i\right]}{[1+\varpi\left(v(s)-v\right)]} = \frac{\mathbb{E}_{s',i|s}\left[u'(e(s',i))\left(1+\frac{\varpi}{2}(v(s')^2-v^2)\right)\right]}{\sum_{s'|s} p(s'|s)}$$

Now use the guessed solution for consumption and the CRRA preferences to write:

$$\mathbb{E}_{s',i|s}\left[\left((1-\vartheta(s'))[(1-\nu(s) - \frac{\varpi}{2\vartheta(s)}\left(v(s)-v\right)^2)\phi^e(s'|s) + \nu(s)r(s')g_i]\right)^{-\sigma} \right. \quad (73)$$

$$\left. \left(\left(1+\frac{\varpi}{2}(v(s')^2-v^2)\right)r(s')g_i - \frac{[1+\varpi\left(v(s)-v\right)]}{\sum_{s'|s} p(s'|s)}\right)\right] = 0$$

With this equation we can solve for $\nu(s)$, which does not depend on either the individual capital or individual shock as long as g_i is iid. Now we need to solve for savings. Using first Foc:

$$\begin{aligned}
u'(e(s,i)) &= \beta \frac{\Pi(s'|s)}{p(s'|s)} \mathbb{E}_i[u'(e(s',i))] \left(1 + \frac{\varpi}{2}(v(s')^2 - v^2)\right); \;\; \forall s, s' \\
u'(e(s,i)) &= \beta \frac{\Pi(s'|s)}{p(s'|s)} \mathbb{E}_i[u'((1-\vartheta(s'))h^e(s',i,k'))] \left(1 + \frac{\varpi}{2}(v(s')^2 - v^2)\right) \\
u'(1-\vartheta(s)) &= \beta \frac{\Pi(s'|s)}{p(s'|s)} \mathbb{E}_i[u'((1-\vartheta(s'))\vartheta(s)o(s',i))] \left(1 + \frac{\varpi}{2}(v(s')^2 - v^2)\right) \\
(1-\vartheta(s))^{-\sigma} &= \beta \frac{\Pi(s'|s)}{p(s'|s)} [(1-\vartheta(s'))\vartheta(s)]^{-\sigma} \mathbb{E}_i o(s',i;\phi^e)^{-\sigma} \left(1 + \frac{\varpi}{2}(v(s')^2 - v^2)\right)
\end{aligned}$$

Where

$$o(s',i;\phi^e) = (1 - \nu(s) - \frac{\varpi}{2\vartheta(s)}\,(v(s)-v)^2)\phi^e(s'|s) + \nu(s)r(s')g_i$$

Using similar manipulations as in the consumer's problem:

$$\left(\mathbb{E}_i o(s',i;\phi^e)^{-\sigma} \left(1 + \frac{\varpi}{2}(v(s')^2 - v^2)\right)\right)^{-1/\sigma} = \frac{\tilde{\beta}(s',s)(1-\vartheta(s))}{(1-\vartheta(s'))\vartheta(s)}; \;\; \forall s, s' \tag{74}$$

Given $\nu(s)$ and $\vartheta(s)$, equation (74) solve for $\phi^e(s'|s)$. Multiplying (74) by $p(s'|s)$ and adding up

$$\sum_{s'} p(s'|s) \left(\mathbb{E}_i o(s',i;\phi^e)^{-\sigma} \left(1 + \frac{\varpi}{2}(v(s')^2 - v^2)\right)\right)^{-1/\sigma} = \sum_{s'} p(s'|s) \frac{\tilde{\beta}(s',s)(1-\vartheta(s))}{(1-\vartheta(s'))\vartheta(s)}; \;\; \forall s$$

Operating with the above equation we obtain:

$$(1-\vartheta(s))^{-1} = 1 + m(s)^{-1} \sum_{s'|s} \left[(\beta\Pi(s'|s))^{1/\sigma} p(s'|s)^{1-1/\sigma} (1-\vartheta(s'))^{-1}\right]$$

Where:

$$m(s) = \sum_{s'} p(s'|s) \left(\mathbb{E}_i o(s',i;\phi^e)^{-\sigma} \left(1 + \frac{\varpi}{2}(v(s')^2 - v^2)\right)\right)^{-1/\sigma}$$

Notice that if $\varpi = 0$ we have the same solution as before. The equation is still linear, with the caveat that $\vartheta(s)$ is included in $m(s)$. With the solution method that we use, this is not a problem. We start guessing $m(s)$ and then we iterate over it.

F.2 Equilibrium

Now **assets market** clearing reads:

$$\phi(s'|s)\zeta(s)x + \phi^e(s'|s)[\vartheta(s)(1-\nu(s)) - \frac{\varpi}{2}(v(s)-v)^2](1-x) = \frac{\omega(s')+h(s')}{W^T(s)};\ \forall s, s' \tag{75}$$

We also have the **goods market clearing** to check:

$$c(s) + e(s) + k'(s) + \frac{\varpi}{2}\left(\frac{k'(s)}{W^e(s)} - v\right)^2 W^e(s) = y(s);\ \forall s$$

$$(1-\zeta(s))W^c(s) + (1-\vartheta(s))W^e(s) + \vartheta(s)\nu(s)W^e(s) + \frac{\varpi}{2}(v(s)-v)^2 W^e(s) = y(s);\ \forall s$$

$$(1-\zeta(s))x + [1-\vartheta(s)(1-\nu(s)) + \frac{\varpi}{2}(v(s)-v)^2](1-x) = \frac{y(s)}{W^T(s)};\ \forall s \tag{76}$$

In theory we should add the individual adjustments costs, but in equilibrium the ratios are all equal, so I avoid the summation to simplify notation.

Recovering $\phi^e(s)$. Once we have $\mathbb{E}_i o(s', i; \phi^e)^{-\sigma}$ how to get ϕ^e? Recall that:

$$\mathbb{E}_i o(s', i; \phi^e)^{-\sigma} = \mathbb{E}_i\left[[(1-\nu(s) - \frac{\varpi}{2\vartheta(s)}(v(s)-v)^2)\phi^e(s'|s) + \nu(s)r(s')g_i]^{-\sigma}\right]$$

Use a second order Taylor approximation around $g_i = 1$ to write:

$$\mathbb{E}_i o(s', i; \phi^e)^{-\sigma} \simeq [o(s', 1; \phi^e)]^{-\sigma}\left[1 + \frac{\sigma(1+\sigma)(\nu(s)r(s'))^2 Var(g_i)}{2o(s', 1; \phi^e)^2}\right]$$

Where $o(s', 1; \phi^e) = o(s', g_i = 1; \phi^e)$. Define:

$$\mathbb{R}(s', s) = \left[1 + \frac{\sigma(1+\sigma)(\nu(s)r(s'))^2 Var(g_i)}{2o(s', 1; \phi^e)^2}\right] \tag{77}$$

and bear in mind that it also depends on V_g and $\phi^e(s', s)$. Therefore:

$$\phi^e(s'|s) = \frac{(\mathbb{E}_i o(s', i; \phi^e)^{-\sigma})^{-1/\sigma}[\mathbb{R}(s', s)]^{1/\sigma}}{1-\nu(s) - \frac{\varpi}{2\vartheta(s)}(v(s)-v)^2} - \frac{\nu(s)}{1-\nu(s) - \frac{\varpi}{2\vartheta(s)}(v(s)-v)^2} r(s') \tag{78}$$

Let $\tilde{q}(s') = 1 + \frac{\varpi}{2}(v(s')^2 - v^2)$. As before, we can expand the term

$$\mathbb{E}_i o(s', i; \phi^e)^{-\sigma} r(s') g_i \simeq r(s') o(s', 1)^{-\sigma}\left[\mathbb{R}(s', s) - \frac{\sigma\nu(s)r(s')}{o(s', 1)} Var(g_i)\right]$$

Using all this in (73) we obtain

$$\sum_{s'|s}\Pi(s',s)\left[r(s')(1-\vartheta(s'))^{-\sigma}o(s',1)^{-\sigma}\tilde{q}(s')\left[\mathbb{R}(s',s)-\frac{\sigma\nu(s)r(s')}{o(s',1)}Var(g_i)\right]\right]=$$
$$\frac{q(s)}{\sum_{s'|s}p(s'|s)}\sum_{s'|s}\Pi(s',s)\left[(1-\vartheta(s'))^{-\sigma}o(s',1)^{-\sigma}\mathbb{R}(s',s)\right]$$

Using equation (74) we get:

$$\sum_{s'|s}\Pi(s',s)\left[r(s')\left(\tilde{\beta}(s',s)\frac{(1-\vartheta(s))}{\vartheta(s)}\right)^{-\sigma}\left[1-\frac{\sigma\nu(s)r(s')}{o(s',1)R(s',s)}Var(g_i)\right]\right]=$$
$$\frac{q(s)}{\sum_{s'|s}p(s'|s)}\sum_{s'|s}\frac{\Pi(s',s)}{\tilde{q}(s')}\left[\left(\tilde{\beta}(s',s)\frac{(1-\vartheta(s))}{\vartheta(s)}\right)^{-\sigma}\right]$$

Doing all the cancellations and replacing $\tilde{\beta}(s',s)^{-\sigma}$:

$$\sum_{s'|s}p(s',s)r(s')\left[1-\frac{\sigma\nu(s)r(s')}{o(s',1)\mathbb{R}(s',s)}Var(g_i)\right]=\frac{q(s)}{\sum_{s'|s}p(s',s)}\sum_{s'|s}\frac{p(s',s)}{\tilde{q}(s')} \tag{79}$$

From (74) we have :

$$\left(\mathbb{E}_i o(s',i;\phi^e)^{-\sigma}\tilde{q}(s')\right)^{-1/\sigma}=\frac{\tilde{\beta}(s',s)(1-\vartheta(s))}{(1-\vartheta(s'))\vartheta(s)}$$

Using the second order Taylor approximation (see (78)), we can write:

$$[1-\nu(s)-\frac{\varpi}{2\vartheta(s)}(v(s)-v)^2]\phi^e(s'|s)+\nu(s)r(s')=\frac{\tilde{\beta}(s',s)(1-\vartheta(s))}{(1-\vartheta(s'))\vartheta(s)}\left[\mathbb{R}(s',s)\tilde{q}(s')\right]^{1/\sigma}$$

Multiplying by $p(s'|s)$ and adding up we obtain:

$$1-\nu(s)-\frac{\varpi}{2\vartheta(s)}(v(s)-v)^2+\nu(s)\sum_{s'|s}p(s'|s)r(s')=\sum_{s'|s}p(s'|s)\frac{\tilde{\beta}(s',s)(1-\vartheta(s))}{(1-\vartheta(s'))\vartheta(s)}\left[\mathbb{R}(s',s)\tilde{q}(s')\right]^{1/\sigma}$$

From equation (79) we obtain:

$$1-\nu(s)-\frac{\varpi}{2\vartheta(s)}(v(s)-v)^2+\nu(s)\sum_{s'|s}p(s'|s)r(s')=$$

$$1 - \nu(s) - \frac{\varpi}{2\vartheta(s)}\left(v(s) - v\right)^2 + \nu(s)\frac{q(s)}{\sum_{s'|s} p(s',s)}\sum_{s'|s}\frac{p(s',s)}{\tilde{q}(s')} + Prem(s)$$

Where:

$$Prem(s) = \sum_{s'|s} p(s',s)\left[\frac{\sigma\nu(s)^2 r(s')^2}{o(s',1)\mathbb{R}(s',s)} Var(g_i)\right] \tag{80}$$

Thus define the ratio $Ra(s',s)$:

$$Ra(s',s) = \frac{\left[\mathbb{R}(s',s)\tilde{q}(s')\right]^{1/\sigma}}{1 + Prem(s) + \nu(s)\left[\frac{q(s)}{\sum_{s'} p(s',s)}\sum_{s'}\frac{p(s',s)}{\tilde{q}(s')} - 1\right] - \frac{\varpi}{2\vartheta(s)}\left(v(s) - v\right)^2} \tag{81}$$

It is clear than when $\varpi = 0$ we are back to the problem before. Thus, the entrepreneur's problem can be solved linearly, given $Ra(s',s)$

$$\frac{1}{(1-\vartheta(s))} = \sum_{s'|s} p(s'|s)\frac{\tilde{\beta}(s',s)}{(1-\vartheta(s'))\vartheta(s)} Ra(s',s) \tag{82}$$

Again, in the algorithm we need to iterate over $Ra(s',s)$.

F.3 State space and laws of motion

Again, it is enough to define s as the pair $\{g_s k, x\}$. To construct $\Pi(s'|s)$ we need the endogenous laws of motions. Start with x, notice that:

$$\begin{aligned}\frac{x(s')}{1-x(s')} &= \frac{W^c(s')}{W^e(s')} = \\ = \frac{\phi(s'|s)\zeta(s)W^c(s)}{\mathbb{E}_i o(s',i,s)\vartheta(s)W^e(s)} &= \frac{\phi(s'|s)\zeta(s)x}{\mathbb{E}_i o(s',i,s)\vartheta(s)(1-x)}\end{aligned}$$

Which can be written as:

$$x(s') = \frac{\phi(s'|s)\zeta(s)x}{\mathbb{E}_i o(s',i,s)\vartheta(s)(1-x) + \phi(s'|s)\zeta(s)x} \tag{83}$$

Using market clearing (75), this equation can also be written as:

$$x(s') = \frac{\phi(s'|s)\zeta(s)}{M(s',s) + \nu(s)\vartheta(s)(1-x)r(s')}x = \frac{\phi(s'|s)\zeta(s)W^T(s)}{W^T(s')}x$$

For the law of motion of $g_s k$ recall that: $k'(s,i) = \nu(s)\vartheta(s)h^e(s,i,k)$ and: $E(s',i,s) = \phi^e(s',s)\vartheta(s)(1-\nu(s))h^e(s,i,k)$. Therefore in every state

$$\frac{E(s',i,s)}{k'(s,i)} = \phi^e(s',s)\frac{(1-\nu(s))}{\nu(s)}$$

Which is independent of i. Thus, assuming that E_1 is also proportional to k

$$k'(s,i) = \nu(s)\vartheta(s)h^e(s,i,k)$$

In short we can write:

$$k'(s) = \nu(s)\vartheta(s)(1-x)W^T(s) \tag{84}$$

Market prices and $\tilde{\beta}(s',s)$. Recall that asset markets clear when

$$\phi(s'|s)\zeta(s)x + \phi^e(s'|s)[\vartheta(s)(1-\nu(s)) - \frac{\varpi}{2}(v(s)-v)^2](1-x) = \frac{\omega(s')+h(s')}{T(s)} = M(s',s)$$

Replacing ϕ and ϕ^e

$$\tilde{\beta}(s',s)\frac{(1-\zeta(s))}{(1-\zeta(s'))}x + \tilde{\beta}(s',s)\frac{(1-\vartheta(s))}{(1-\vartheta(s'))}\left[\mathbb{R}(s',s)\tilde{q}(s')\right]^{1/\sigma}(1-x)$$
$$-\nu(s)r(s')\vartheta(s)(1-x) = M(s',s)$$

Therefore:

$$\tilde{\beta}(s',s) = \frac{M(s',s)+\nu(s)r(s')\vartheta(s)(1-x)}{\left(\frac{(1-\zeta(s))}{(1-\zeta(s'))}x + \frac{(1-\vartheta(s))}{(1-\vartheta(s'))}\left[\mathbb{R}(s',s)\tilde{q}(s')\right]^{1/\sigma}(1-x)\right)} \tag{85}$$

Then, we can recover the prices using the fact that:

$$p(s'|s) = \beta\Pi(s'|s)\tilde{\beta}(s',s)^{-\sigma} \tag{86}$$

Notice that

$$M(s',s)+\nu(s)r(s')\vartheta(s)(1-x) = M(s',s)+r(s')K(s') = \frac{W^T(s')}{W^T(s)}$$

Get ϕ^e from market clearing. Notice that multiplying (75) by $p(s',s)$ and adding up we obtain:

$$\zeta(s)x + [\vartheta(s)(1-\nu(s)) - \frac{\varpi}{2}(v(s)-v)^2](1-x) = \frac{h(s)}{W^T(s)}$$

Thus,

$$[\vartheta(s)(1-\nu(s)) - \frac{\varpi}{2}\left(v(s)-v\right)^2] = \frac{1}{1-x}\left[\frac{h(s)}{W^T(s)} - \zeta(s)x\right]$$

Using the last in (75) we obtain

$$\phi(s'|s)\zeta(s)x + \phi^e(s'|s)\left[\frac{h(s)}{W^T(s)} - \zeta(s)x\right] = \frac{\omega(s') + h(s')}{W^T(s)}$$

Therefore, it is exactly the same as with $\varpi = 0$:

$$\phi^e(s'|s) = \frac{\omega(s') + h(s') - \phi(s'|s)\zeta(s)xW^T(s)}{h(s) - \zeta(s)xW^T(s)} \tag{87}$$

Get $\nu(s)$ from market clearing. Recall, at this point ϑ is supposed to be known. We can use (76):

$$(1-\zeta(s))x + [1 - \vartheta(s)(1-\nu(s)) + \frac{\varpi}{2}\left(v(s)-v\right)^2](1-x) = \frac{y(s)}{W^T(s)} = \tilde{y}(s)$$

Since $v(s) = \nu(s)\vartheta(s)$, now it is quadratic in ν (it is linear when $\varpi = 0$)

$$\nu^2\vartheta\frac{\varpi}{2} + \nu(1-\varpi) = \frac{\tilde{y} - 1 + \zeta x}{(1-x)\vartheta} - \frac{\varpi}{2\vartheta}v^2 + 1$$

The solutions are:

$$\nu = \frac{1}{\varpi\vartheta}\left[-(1-\varpi) \pm \sqrt{(1-\varpi)^2 + 2\varpi\vartheta\left(\frac{\tilde{y} - 1 + \zeta x}{(1-x)\vartheta} - \frac{\varpi}{2\vartheta}v^2 + 1\right)}\right]$$

For $\varpi < 1$ only the positive root can be a solution, therefore:

$$\nu(s) = \frac{1-\varpi}{\varpi\vartheta(s)}\left[-1 + \sqrt{1 + \frac{2\varpi}{(1-\varpi)^2}\left(\frac{\tilde{y}(s) - 1 + \zeta(s)x}{(1-x)} - \frac{\varpi}{2}v^2 + \vartheta(s)\right)}\right] \tag{88}$$