The Macroeconomics of Hedging Income Shares

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Abstract

The debate about the falling labor share brought attention to the income shares trends, but less attention has been devoted to their variability. We analyze how their fluctuations can be insured between workers and capitalists, and the corresponding implications for financial markets. We study a neoclassical growth model with aggregate shocks that affect income shares and financial frictions that prevent firms from fully insuring idiosyncratic risk. We examine theoretically how aggregate risk sharing is shaped by the combination of idiosyncratic risk and moving shares. Accumulation of safe assets by capitalists and risky assets by workers emerge naturally as a tool to insure income shares' risk. We calibrate the model to the U.S. economy and show that low interest rates, rising capital shares, and accumulation of safe assets by firms and risky assets by households can be rationalized by persistent shocks to the labor share.

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1 Introduction

For several decades the ubiquity and the robustness of the Kaldor facts led to the dominant belief that capital and labor income shares are roughly constant over time. An important implication of this paradigm is the impossibility of insurance between workers and capitalists. Since aggregate shocks affect both agents equally, even if markets existed, aggregate risk would be uninsurable. However, many recent studies find that income shares are moving far more than in Kaldor's original predictions.[1] This opens new possibilities: if aggregate shocks have different impacts on capitalists and workers, they can be insured. If so, many questions arise: How do these insurance possibilities affect the financial markets? Which kinds of assets could be affected? Finally, how quantitatively important are the implications?

How income shares are insured is shaped by their stochastic properties. Many studies show that the labor share is pro-cyclical in the short run and counter-cyclical in the medium-long run.[2] However, the pro-cyclicality is short-lived, tapering off after approximately a year. For this reason, we focus on the more relevant medium-long run (although our theory does not depend on it). We first show theoretically that a counter-cyclical labor share can be insured between capitalists and workers by capitalists accumulating risk-free assets and lending them to workers, who use loans to leverage and buy risky assets. Next, we analyze business cycle dynamics, which yields interesting predictions. Upward changes in the capital share reduce capitalists' risk absorption, hindering aggregate risk sharing. As a consequence, the demand for precautionary savings increases, decreasing the risk-free interest rate and increasing the risk premium. Finally, we show that these qualitative predictions are also quantitatively sizable.

The channel that we analyze is simple and intuitive and has been overlooked despite being consistent with several seemingly unconnected findings. There is growing literature trying to explain what is known as the *Corporate Saving Glut*, shifting the view of corporations from net borrowers to net lenders. Our theory generates this fact and has the additional implication that the *Corporate Saving Glut* is accompanied by a *Households' Equity Glut*. Indeed, they are two sides of the same coin: optimal portfolio choices to insure income shares. Furthermore, the theory we present is also consistent with widely documented movements in asset prices, including a continuously falling interest rate and an increasing risk premium[3].

[1] See, for instance, Karabarbounis and Neiman (2014) and Rodriguez and Jayadev (2010).

[2] See Ríos-Rull and Santaeulàlia-Llopis (2010), León-Ledesma and Satchi (2018), and Cantore et al. (2019) for estimations and potential explanations.

[3] See Del Negro et al. (2017b) for the U.S. and Del Negro et al. (2017a) for the global trend in the interest

We build on the neoclassical growth model, allowing for income shares that fluctuate persistently over time. The economy is populated by a continuum of capitalists with different endowments of capital and workers who supply labor inelastically. Capitalists rent labor and carry out the production. Workers consume and fund firms through the financial markets, but do not own capital directly. There is a contracting friction; as in DeMarzo and Fishman (2007), the capitalists' returns cannot be verified as they can privately divert resources for consumption. Firms would like to pool the idiosyncratic risk and obtain funding, but they are subject to a "skin in the game" constraint: the lenders force firms to keep a fraction of their investment. Nevertheless, there are enough financial instruments available such that both capitalists and workers can perfectly insure against aggregate risk. Yet, the contracting friction prevents capitalists from fully insuring the idiosyncratic risk, which affects the agents' willingness to bear aggregate risk.[4]

The key departure from the literature is that we move away from constant shares technologies (Cobb-Douglas or AK). Since we are focusing on medium-long time horizons, we assume that the labor share is counter-cyclical (the capital share is pro-cyclical) so that capitalists benefit more in booms and suffer more in recessions. We want to stress that our purpose is to analyze how income shares exogenous fluctuations affect the financial markets, not to characterize these markets fully. There are many important factors that we abstract from that are undoubtedly relevant, if not the most important. Nevertheless, as long as the capitalist's and workers' income shares fluctuate over the business cycle, there is something to insure and, thus, affect financial markets.

We begin by characterizing asset prices and quantities in a simplified two-period economy and then extend the results to richer infinity horizon economy. The simple environment helps to understand the main trading patterns of financial assets and how the presence of idiosyncratic risk is key generating the observed financial market's trends stressed in this paper. To do so, we assume that agents have access to a complete set of Arrow-Debreu (AD) securities contingent on the realization of the aggregate shock, but not on the realization of the idiosyncratic one.

In Proposition 1, we show that to insure aggregate risk, workers and capitalists trade AD securities where, for instance, if the capital share increases, capitalists compensate workers with contingent transfers, and vice versa. However, the predicted trends crucially depends on the presence of idiosyncratic risk. When such risk does not exist the economy is memory-less and trend-less: the shock's realizations or their history do not

rate. Both papers attribute most of the fall in the risk-free rate to an increase in the convenience yield.

[4] We assume that workers are not subject to idiosyncratic risk. Thus, the fact that capitalists are exposed to id. risk must be interpreted in relative terms throughout the paper. There is ample evidence that firms are more exposed to idiosyncratic risk than workers, see for example Guiso et al. (2005).

affect consumption, AD positions or asset prices patterns. In contrast, the introduction of idiosyncratic risk creates an additional capitalists' demand for precautionary savings. They satisfy this demand by moving their positions in AD securities away from the full insurance of income shares. As a result, aggregate shocks *change the relative wealth*, generating further portfolios rebalancing: the economy now is history dependent. We call this additional channel the "wealth effects" channel. We also characterise the enduring effect of aggregate shocks, stating well defined testable empirical predictions. We show that *larger capital shares are associated to lower risk-free rates, larger risk premia and increased capitalists's positions on AD securities*. Intuitively, a higher capitalists' share of total output increases profits but magnifies their total variace, as profits are subject to idiosyncratic risk. Hence, states of higher capital shares are also states with higher idiosyncratic risk, leading to higher demand for insurance, which in turn delivers the result.

We then show that, when there are two possible aggregate shocks, the optimal insurance contract can be implemented using only a risk-free and a risky asset. This equilibrium is implemented by firms taking a long position on the risk-free asset (saving) and households taking a long position on the risky asset (buying equity). Because markets must clear, a positive position by one sector in a given asset implies a negative position by the other. Intuitively, workers leverage (borrow from firms) and buy shares to participate in the capital share changes.

This market allocation is reminiscent of a corporate savings glut, which is complemented by a *households' equity glut*. If there were not idiosyncratic risk the financial positions and asset prices would be constant and independent of history. When there is idiosyncratic risk the tendency towards capitalists accumulating risk-free assets and workers accumulating equity remains. But the wealth effects create an additional channel that amplifies portfolio rebalancing over time. As the capital share increases so it does the demand for precautionary savings, generating an increase in the firms' long positions on risk-free assets. In parallel, workers increase their leverage, borrowing from capitalists to increase their equity holdings. All this happens while the risk-free rate is falling and the equity premium increasing.

Given the "low variance" of income shares, one may be concerned that these predictions, although theoretically interesting, are quantitatively irrelevant. To quantify magnitudes, we calibrate an otherwise standard economy, first by using usual parameter values, and then by replicating standard moments and evaluate its performance in terms of financial quantities and prices. We find that a labor share variance of 0.5%, half of what is observed, implies that workers ought to borrow around 1.6GDP and hold equity at around 0.8GDP. This happens with a risk-free rate of 1% or less (depending on the labor share) and with an equity premium which is between 5% and 6%. Comparing these results to the U.S. economy, households held around 1.3GDP in equity in 2018, while total private debt was also around 1.3GDP. Even though we do not target any of the financial markets' moments, and we do not include any other friction and/or motive for trading, we obtain strikingly reasonable quantities. Not only that, the generated moments for the risk-free rate and risk premium are in line with most estimations.

The paper is organized as follows. Section 1.1 reviews the literature. Section 2 highlights the main mechanisms in a tractable two period model. In Section 3, we present a general model and generalize most results. In Section 4 we calibrate and evaluate numerically the general model. The conclusion follows. All proofs are in Appendices.

1.1 Literature review.

This paper is motivated by the recent literature emphasizing changes in the labor share. Since Karabarbounis and Neiman (2014), several studies have pointed out the apparent labor share downward trend. The potential reasons for this trend range from a fall in the price of investment, the growing importance of housing (Rognlie, 2015), rising mark-ups (De Loecker et al., 2020 and Barkai, 2020), demographics (Hopenhayn et al., forthcoming), to the possibility that the labor share is not falling and it is just a measurement issue (Koh et al., 2020).[5] In this paper *we take the changes in the labor share as exogenous*.[6] This reflects the fact that in our analysis it is not important why the labor share is changing as long as it fluctuates. Moreover, we abstract from the potential feedback from the asset markets to the income shares. Finally, we do not focus on the potential existence of a downward trend, but rather on its cyclical properties. This dimension has been mostly overlooked in the literature except, to the best of our knowledge, by Ríos-Rull and Santaeulàlia-Llopis (2010), León-Ledesma and Satchi (2018) and Cantore et al. (2019).

The positive implications of our paper relate to the recent literature that connects low risk-free rates, risk-premia, and changes in the labor share. Caballero et al. (2017) proposes an accounting framework that connects falling short term real rates, a constant marginal product of capital, the labor share decline, and a stable earnings yield from corporations. In contemporaneous works Eggertsson et al. (2021) and Farhi and Gou-

[5]Since Koh et al. (2020) there has been a debate about whether the labor share is falling or not, and if so, to which extent. While the downward trend seems to be robust in the U.S., the labor share appears to be stationary in the rest of the world. See Gutiérrez and Piton (2020).

[6]Grossman et al. (2017) argues in favor of a response to declining aggregate productivity when human and physical capital are complements. Oberfield and Raval (2014) find that the elasticity of capital and labor in the U.S. manufacturing sector has been stable around a value that is substantially lower than the one implied by previous estimates.

rio (2018) document and link the simultaneous patterns of a decreasing labor share and risk-free rates with an increasing savings supply and risk premia. Eggertsson et al. (2021) argue that these trends are mostly due to rising markups.[7] In contrast, Farhi and Gourio (2018) use a different methodology and find that even though mark-ups could be playing an important role, it is the risk premia and unmeasured intangibles the key. Similarly Chirinko and Mallick (2022) find that the private return in capital has been increasing globally and argue that 60% of the increase can be explain by a larger risk premia. In our paper the mechanism generating these facts is completely different. There is no technological or competition factor; all the trends arise due to financial trading to insure income shares. Also Chen et al. (2017) document the corporate savings glut in the U.S. and relate it to the labor share decline. They argue that it is driven by a combination of changes in the real interest rate, the price of investment goods, corporate income taxes and the increase in markups. In our setup the interest rate is endogenous and, hence, it is not a cause but another implication of the theory.

Our paper is also related to the literature on the financial amplification of aggregate shocks, following the seminal work of Bernanke and Gertler (1989) and Kiyotaki and Moore (1997). We build on the recent contributions of He and Krishnamurthy (2012), Brunnermeier and Sannikov (2014) and Di Tella (2017), where financial frictions and heterogeneity play a key role. We depart from the previous studies by introducing human capital and income shares correlated with the business cycle. These two assumptions allow us to study positive and normative implications of changes in labor and capital shares over the business cycle. Bocola and Lorenzoni (2020), building on Krishnamurthy (2003), also introduce labor income, but with constant income shares. One of their objectives is to explain why financial intermediaries hold so much risk that is amplified once aggregate shocks are realized. In our work, instead, optimal insurance contracts and constant shares do not generate amplification of aggregate shocks; there is amplification only with variable income shares. The difference with respect to Bocola and Lorenzoni (2020) is the modeling of the financial friction, which they assume is state dependent collateral constraint, while in our paper is a "skin in the game" constraint. Carvalho et al. (2016) and Auclert et al. (2019) provide a real explanation for low interest rates based on demographics. The channel through which demographics imply a lower interest rate is that a longer life span implies a higher supply of safe assets for retirement, and lower demand for investment. Our paper focuses on changes in the labor share and in idiosyncratic risk

[7] Nekarda and Ramey (2020) includes a detailed discussion of the empirical literature on markups over the business cycle. In addition, they show that the markup with respect to TFP shocks is pro-cyclical when proxied via the output labor ratio, and a-cyclical and counter-cyclical when proxied via the output-capital ratio.

that increase firms' precautionary savings, which in turn depresses the real rate.[8]

2 Hedging income shares

In this section we study a two-period economy with exogenous capital to obtain a sharp characterization of the implications on asset prices and quantities of changes in income shares. In Section 3 we develop an infinite horizon economy with endogenous investment to: (a) show that all findings in this section hold in the general model, and (b) explore the quantitative implications.

2.1 Simplified environment

There are two types of agents: workers and capitalists. The economy lasts for two periods, $t = 1, 2$. There are two sources of uncertainty: aggregate shocks, indexed by $s \in \mathbb{S}$ and idiosyncratic production shocks, indexed by $i \in \mathbb{I}$, which occur with probability $\Pi(s, i)$. In this section, for simplicity, we assume that there is no time discounting.

Workers. Workers are endowed with initial assets A_1 and can supply one unit of labor at no utility cost. Labor income in period one is certain and given by ω_1, which denotes the wage rate. In period two they receive $\omega(s)$ as labor income, which is contingent on the realization of the aggregate shock. To insure against variations in wages, workers have access to a complete set of Arrow-Debreu (AD) securities, denoted by $A_2(s)$, contingent on state s. Each asset can be traded at prices $p(s)$.

Notice that we allow for as many aggregate financial assets as possible aggregate states. There are two reasons for this modelling choice. First, it is a standard setup in the literature (e.g., He and Krishnamurthy (2012), Di Tella (2019), Di Tella (2017), among others), so we can easily relate our findings to previous contributions. Second, it is a reasonable approximation of reality, in which there are multiple types of financial assets, whose payoffs do not rely on nor are constrained by individual moral hazard or commitment problems, and help workers and entrepreneurs insure aggregate shocks. By properly combining them, one can replicate the same allocations as with AD securities. securities.[9] Whether complete insurance is achieved or not ultimately depends on the

[8]The effects of a low real interest rate can be amplified by nominal frictions. The key idea of secular stagnation, proposed by Hansen (1939), is that the real rate needed to achieve full employment is negative, casting on the economy shadows of low growth and high unemployment. See for instance, Eggertsson et al. (2019), Benigno and Fornaro (2018), Schmitt-Grohé and Uribe (2012) and Marx et al. (2019).

[9]In discrete time models, it is well known that to complete the market there must be as many non-state contingent assets,with imperfectly correlated prices, as possible states. When the time interval becomes

assets' prices.

The worker maximizes expected utility:

$$\max_{\{c_1, c_2(s), A_2(s)\}} u(c_1) + \mathbb{E}_s(u(c_2(s)))$$

$$s.t. \quad c_1 + \sum_s p(s) A_2(s) \leq A_1 + \omega_1 \tag{1}$$

$$c_2(s) \leq A_2(s) + \omega_2(s) \tag{2}$$

The worker uses initial assets A_1 and income ω_1 to consume and buy AD securities. In the second period consumption is given by the income realization and the payoff of the assets acquired in the first period.

Capitalists. Firm's owners are endowed with initial financial assets E_1 and exogenous capital income $\{\pi_1, \pi_2(s,i)\}$, which is a function of aggregate and idiosyncratic shocks. We start by assuming that capital income is exogenous in order to highlight the insurance mechanism. In Section 3 we allow capitalists to invest in physical capital, which generates additional real effects.

Capitalists would like to share the idiosyncratic risk, but are prevented from doing so due to a financial friction: they could divert income to a private account. As a result, they must retain some idiosyncratic risk. This feature can be rationalized as the result of an optimal risk-sharing contract between the entrepreneur, with moral hazard, and a principal (the market), as in DeMarzo and Fishman (2007) and Di Tella (2017).[10] Capitalists can buy a complete set of AD securities $E(s)$, which are contingent on s but not on i. The capitalist's problem is:

$$\max_{\{e_1, e_2(s,i), E_2(s)\}_{s \in S}} u(e_1) + \mathbb{E}_{s,i}(u(e_2(s,i)))$$

infinitesimal and the underlying risk is characterized by a Brownian motion (in continuous time), only two assets are needed. See for example Merton (1992).

[10]Because of moral hazard, the optimal contract provides only partial insurance of idiosyncratic risk: entrepreneurs must keep some "skin in the game". See the online Appendix E and Section 3.2 for additional details on the optimal contract for the entrepreneur. On the other hand, due to the lack of contracting friction on the side of consumers, the consumers who hold equity in the firm can diversify the risk by pooling their ownership, and for that reason, they do not hold id. risk.

$$s.t. \quad e_1 + \sum_s p(s) E_2(s) \leq E_1 + \pi_1$$
$$e_2(s,i) \leq E_2(s) + \pi_2(s,i)$$

for all (s,i). The capitalist can use initial assets E_1 to consume and buy AD securities. In the second period, consumption is given by the realization of the return to capital, $\pi_2(s,i)$, and the payoff of the assets acquired in the first period, $E_2(s)$.

Profits and wages. Profits and wages are given by

$$\pi(s,i) = g_i \alpha(s) Y(s) \tag{3}$$
$$\omega(s) = (1 - \alpha(s)) Y(s) \tag{4}$$

where $g_i > 0 \; \forall i$ and $\mathbb{E}(g_i) = 1$. Equations (3) and (4) stress the sources of income variations. In addition to the capitalist's exposure to idiosyncratic risk, capitalists and workers income will vary after an aggregate shock. The shock changes both aggregate output and the relative claims to it. In the quantitative section we generate time-varying income shares with a CES production function and shocks to the capital quality.

Markets. Market clearing implies:

$$c_1 + e_1 = Y_1 \tag{5}$$
$$c_2(s) + \mathbb{E}_i(e_2(s,i)) = Y_2(s) \qquad \forall s \tag{6}$$
$$A_2(s) + E_2(s) = 0 \qquad \forall s \tag{7}$$

where $Y_1 = \pi_1 + \omega_1$ and $Y_2(s) \equiv \int y_2(s,i) di$, $\forall s$. The first constraint, equation (5), is market clearing for goods in period 1. It also implies that the initial asset holdings are such that $A_1 + E_1 = 0$. The second constraint, equation (6), is market clearing for goods in period 2. Note that the idiosyncratic *i.i.d.* shocks cancel out in the aggregate. The final constraint, equations (7), specifies that asset markets clear.

Definition: A *Competitive Equilibrium* is a consumption allocation $\{c_1, e_1, c_2(s), e_2(s,i)\}_{s \in S}^{i \in \boldsymbol{I}}$, asset holdings $\{A_2(s), E_2(s)\}_{s \in S}$ and asset prices $\{p(s)\}_{s \in S}$ such that: *(i)* given prices the worker maximizes utility by choosing asset holdings and consumption; *(ii)* given prices the capitalist maximizes utility by choosing financial asset holdings and consumption; and *(iii)* markets clear.

2.2 Equilibrium Characterization

We now derive the optimality conditions for workers and capitalists. From the individual problems' first-order conditions we obtain:

$$
\begin{aligned}
p(s)u'(c_1) &= \Pi(s)u'(c_2(s)) \\
p(s)u'(e_1) &= \Pi(s)\mathbb{E}_i[u'(e_2(s,i))].
\end{aligned}
$$

A key element of the above equations is that, due to the existence of a complete set of AD securities for the aggregate state, the Euler equations hold state by state. The two first-order conditions together imply:

$$
\frac{u'(e_1)}{u'(c_1)} = \frac{\mathbb{E}_i[u'(e_2(s,i))]}{u'(c_2(s))} \tag{8}
$$

for all s. Note that equation (8) states that the ratio of future average marginal utilities is constant across states. Define the holding of state s AD securities as a fraction of state s output as $\phi(s) := \frac{A_2(s)}{Y_2(s)}$. Market clearing implies that:

$$
\begin{aligned}
A_2(s) &= \phi(s)Y_2(s) \\
E_2(s) &= -\phi(s)Y_2(s).
\end{aligned}
$$

Then, from market clearing in the goods market in the first period, and assuming CRRA preferences with parameter σ, we can rewrite (8) as:

$$
\frac{u'(e_1)}{u'(c_1)} = \frac{\mathbb{E}_i[(-\phi(s)Y_2(s) + \alpha(s)Y_2(s)g_i)^{-\sigma}]}{(\phi(s)Y_2(s) + (1-\alpha(s))Y_2(s))^{-\sigma}}. \tag{9}
$$

For future reference define the worker's wealth share in period's 1 and 2 as:

$$
\begin{aligned}
x_1 &= \frac{A_1 + \omega_1 + \sum_s p(s)\omega_2(s)}{Y_1 + \sum_s p(s)Y(s)}, \\
x_2(s) &= \frac{A_2(s) + \omega_2(s)}{Y(s)}.
\end{aligned} \tag{10}
$$

The numerator is the worker's total wealth, and the denominator is the economy's total wealth. In period 1, the worker's wealth is the initial assets plus the present value of wages. Total wealth in the economy is the current plus the present value of future total output. For the second period, the share of total wealth may depend on the aggregate shock. Note that both shares are endogenous and determined in equilibrium. In the next section, we characterize the equilibrium as a function of the wealth share and discuss the conditions under which it is constant. We say that there is an amplification of aggregate shocks whenever the share of wealth is state-dependent.

2.3 Positive implications

Define the "certainty equivalent" $g^{ce}(\alpha, \phi; s)$ as the function satisfying:

$$(-\phi(s) + \alpha(s)g^{ce}(\alpha, \phi))^{-\sigma} = \mathbb{E}_i[(-\phi(s) + \alpha(s)g_i)^{-\sigma}] \quad \forall s \in \mathbb{S}. \tag{11}$$

This function depends on $\alpha(s)$, which is a primitive of the problem, and ϕ, which is the choice of AD securities. Note that, because for CRRA preferences marginal utility is convex (i.e. $u''' > 0$), and as a result $g^{ce}(\alpha, \phi; s) \leq 1 \; \forall s$, with equality only if $\mathbb{V}ar(g_i) = 0$.[11] Define also $g_s := \frac{Y(s)}{Y_1}$. The main result of the subsection, Proposition 1, characterizes the asset prices and asset holdings in the Competitive Equilibrium.

Proposition 1. *If $\mathbb{V}ar(g_i) > 0$, and $\mathbb{V}ar(\alpha(s)) > 0$, then, in a competitive equilibrium:*

(a) Prices and financial positions satisfy:

$$p(s) = \Pi(s)\left[1 + \alpha(s)(g^{ce}(s) - 1)\right]^{-\sigma} g_s^{-\sigma}, \tag{12}$$
$$\phi(s) = x_1 - (1 - \alpha(s)) + \alpha(s)(g^{ce}(s) - 1)[x_1 + \Gamma(g^{ce})] + \Gamma(g^{ce}), \tag{13}$$

and for $\Gamma(\cdot)$:$\mathbb{R} \to \mathbb{R}_+$, which is determined in equilibrium, it holds that $\Gamma(1) = 0$.

(b) Wealth shares evolve according to:

$$x_2(s) - x_1 = \alpha(s)(g^{ce}(s) - 1)[x_1 + \Gamma(g^{ce})] + \Gamma(g^{ce}). \tag{14}$$

(c) Moreover, let $p(s)^{CM}$ and $\phi(s)^{CM}$ be the solutions to (12) and (13) when $\mathbb{V}ar(g_i) = 0$. Then, there are precautionary savings:

$$-\sum_s p(s)\phi(s) > -\sum_s p(s)^{CM}\phi(s)^{CM}.$$

[11] With an abuse of terminology we will refer to g^{ce} as the "certainty equivalent" even though we are not working with utilities but with marginal utilities. Technically speaking, any utility function that has a positive coefficient of *prudence* would generate the same outcome. Further, equation (11) also points out the relevance of the minimum realization of the idiosyncratic shock, i.e. $\underline{g}_i$. If $\underline{g}_i = 0$, only solutions with $\phi(s) \leq 0$ are admissible, and thus the entrepreneur cannot borrow. Alternatively, if $\underline{g}_i = 1$, the entrepreneur can borrow up to the full expected value of future income. For the remainder of the paper, we assume that $\underline{g}_i > 0$ is sufficiently large such that both borrowing and lending are feasible in equilibrium.

Proof. See Appendix C.2.

We will now analyze the different cases of Proposition 1 depending on whether the market is complete or incomplete, and whether the income shares are varying or not. Discussing these cases will help us to build intuition of the forces that drive the allocations in the infinite horizon model of Sections 3 and 4.

Complete markets: $\mathbb{V}(g_i) = 0$**.** When there is no idiosyncratic risk, Proposition 1 characterizes the insurance arrangement to hedge aggregate risk. Since $\mathbb{V}ar(g_i) = 0$ implies $g^{ce}(s) = 1$ and therefore $\Gamma(1) = 0$, prices and asset holdings given by equations (12) and (13) for each s simplify to:

$$
\begin{aligned}
p^{CM}(s) &= \Pi(s) g_s^{-\sigma}, &\quad (15)\\
\phi^{CM}(s) &= x_1 - (1 - \alpha(s)), &\quad (16)\\
x_2(s) &= x_1, &\quad (17)
\end{aligned}
$$

There are three features of this allocation that are noteworthy. First, regarding asset prices, the fact that prices are given by (15) is a standard result in a Lucas (1978) economy. The price to transfer consumption to states that are more likely or feature lower endowments is higher. More importantly, because agents can fully share risk, the price of the state s security depends on the aggregate endowment, but not on its distribution.

Second, regarding asset holdings given by (16), note that if the income shares are constant, then there is no need for insurance between workers and entrepreneurs. Both types of agents are equally hit by aggregate shocks, and for that reason, $\phi(s)$ is constant. Alternatively, when income shares are stochastic, because aggregate shocks affect differently profits and wages, there are gains from trade in financial assets. For example, from (16), we can observe that workers will buy insurance against states in which the capital (labor) share is higher (lower). For further reference note that the positions in AD securities, $\phi(s)^{CM}$, move one to one with the realization of $\alpha(s)$.

Third, regarding the evolution of the wealth shares, from equation (17) it follows that they are constant. We denote it as $x^{CM} := x_2(s) = x_1$ for all s, which is (10) evaluated at the complete market prices, which are given by (15).[12] Hence, the wealth shares are constant over aggregate states and across periods, which is an expression of full insurance. Intuitively, capitalists fully compensate workers with contingent payments when the capital income share increases and vice versa. This compensation through AD securities is such that both types of agents consume a constant proportion of the aggregate

[12] Recall that $g^{ce}(s) = 1$ and $\Gamma(1) = 0$.

resources, which is independent of the current income shares and the history of shocks: the economy is memory-less.

Incomplete markets: $\mathbb{V}(g_i) > 0$**.** Things are different when capitalists are subject to idiosyncratic risk. First, regarding prices, from equation (12) and (15) it is evident that $p(s) > p^{CM}(s)$ for all s, as long as there is idiosyncratic risk, which implies $g^{ce}(s) < 1$ for all s. How different these prices are depends on the factor $[1 + \alpha(s)(g^{ce}(s) - 1)]^{-\sigma}$, which is increasing in $\alpha(s)$. A larger capital share realization in state s implies that capitalists bear more idiosyncratic risk in that state, since the variance of profits is increasing in both output and the capital share.[13] For this reason the capitalist wants to increase its insurance against the realization of that state, and as a result, insuring aggregate risk becomes more expensive. This is one of the key intuitions of this paper.

Second, from (13), we can observe that there are opposing forces regarding asset positions. Recall that $E(s) = -\phi(s)Y(s)$, hence the capitalist increases her position in state s if and only if $-\phi(s) > -\phi(s)^{CM}$. This happens whenever:

$$\phi(s)^{CM} - \phi(s) \;\; = \;\; x_1^{CM} - x_1 - \alpha(s)(g^{ce}(s) - 1)[x_1 + \Gamma(g^{ce})] - \Gamma(g^{ce}) > 0.$$

We show in Appendix C.4 that $x_1^{CM} > x_1$. Intuitively, due to the higher price of insurance, the worker is relatively poorer: the Net Present Value (NPV) of total output increases more than the worker's wealth NPV. The capitalist, as we just mentioned, demand insurance. And because the capitalist has higher relative wealth with incomplete markets, demands relatively more insurance. The second term, $-\alpha(s)(g^{ce}(s) - 1)[x_1 + \Gamma(g^{ce})]$, is also positive because $g^{ce}(s) < 1$. This is a hedging demand due to idiosyncratic risk. Finally, the last term is negative. Unlike the previous term, which depends on s this last term captures the "average" shift in the demand for AD securities. Quantitatively, the first two terms dominate. As a result, capitalists increase asset positions overall, with a more pronounced increase on states with higher capital share $\alpha(s)$.

Intuitively, capitalists move away from the full insurance of aggregate risk to accumulate precautionary savings and hedge against idiosyncratic risk. As we discussed before, states with higher capital shares also feature higher variance of profits, to insure part of that risk capitalists buy insurance. The insurance is imperfect, though, because they still need to bear idiosyncratic risk. Because the insurance is imperfect, wealth shares vary as a result of changes in the income shares, as is apparent from (14). Moreover, part (c) shows that the total amount spent on insurance by capitalists is now higher, which is intuitive.

[13] The variance of profits given idiosyncratic risk is $\mathbb{V}(\pi(s,i) \mid g_i) = g_i^2[\alpha(s)^2\mathbb{V}(Y(s)) + Y(s)^2\mathbb{V}(\alpha(s)) + 2Cov\,(\alpha(s), Y(s))]$ and $\mathbb{V}(\pi(s,i) \mid s) = \mathbb{V}(g_i)\alpha(s)^2Y(s)^2$.

These findings stress the relevance of incorporating idiosyncratic capital income risk. First, it diminishes the power of the income shares insurance motive and, as it is evident from equation (14), the economy is no longer memoryless. Both elements would be important in our quantitative analysis where we analyse the quantitative power of theory and the potential history dependent effects of income shares shocks. For the remaining of the paper we call this additional channel "wealth effects," i.e., the induced changes in quantities and prices due to movements in the wealth share x.

2.4 Implementation with two assets

Proposition 1 helps us to build intuition regarding allocations. However, it is instructive, and useful for empirical analysis, to map those predictions to assets which are observed in reality. In Proposition 2 we study a decentralization with two shocks and two assets. In particular, suppose there are only two aggregate states $s_L < s_H$, and two financial assets, a risk-free bond B and a stock-market-indexed risky asset A with payoffs $A \times \pi_2(s)$ for $s = L, H$. The (gross) risk-free rate is denoted by R and P_A denotes the price of the risky asset. We focus on the case in which the labor share is pro-cyclical; i.e., $\alpha(H) > \alpha(L)$ and $Y_2(H) > Y_2(L)$. $\{A^w, B^w\}$ is the portfolio allocation of workers and $\{A^e, B^e\}$ is the capitalists' aggregate portfolio allocation.

Proposition 2. *The position in risk free debt and equity of the worker is given by:*

$$R_L B^w = -\left(\frac{\alpha(H)-\alpha(L)}{\pi_2(H)-\pi_2(L)}\right) Y_2(L)Y_2(H)(1-x_1) - \frac{Y_2(L)Y_2(H)\alpha(H)\alpha(L)}{\pi_2(H)-\pi_2(L)}\left(g^{CE}(H)-g^{CE}(L)\right)x_1 + \mathbf{\Psi} \tag{18}$$

where $\mathbf{\Psi} = \Gamma \times \frac{\alpha(H)\alpha(L)\left(g^{CE}(L)-g^{CE}(H)\right)+\alpha(H)-\alpha(L)}{(Y_2(L)Y_2(H))^{-1}(\pi_2(H)-\pi_2(L))}$.

$$A^w = 1 - \left(\frac{Y_2(H)-Y_2(L)}{\pi_2(H)-\pi_2(L)}\right)(1-x_1) + x_1\left[\frac{\alpha(H)Y_2(H)g^{CE}(H)-\alpha(L)Y_2(L)g^{CE}(L)}{\pi_2(H)-\pi_2(L)} - 1\right] + \mathbf{\Xi} \tag{19}$$

$\mathbf{\Xi} = \Gamma \times \frac{Y_2(H)\left(\alpha(H)\left(g^{CE}(H)-1\right)+1\right)-Y_2(L)\left(\alpha(L)\left(g^{CE}(L)-1\right)+1\right)}{\pi_2(H)-\pi_2(L)}$.

Proof. See Appendix C.3. □

As we did in the previous subsection, we begin by discussing the case of complete markets, which will help us understand the case with incomplete markets. In the case of *complete markets* equations (18) and (19) become:

$$R^{CM}B^{CM} := -\left(\frac{\alpha(H)-\alpha(L)}{\pi_2(H)-\pi_2(L)}\right)Y_2(L)Y_2(H)\left(1-x_1^{CM}\right), \tag{20}$$

$$A^{CM} := 1-\left(\frac{Y_2(H)-Y_2(L)}{\pi_2(H)-\pi_2(L)}\right)\left(1-x_1^{CM}\right), \tag{21}$$

where x_1^{CM} is the wealth ratio evaluated at $p^{CM}(s)$. First, notice that workers take an active position on the risk-free asset only if $\alpha(H) \neq \alpha(L)$. In particular, with a Cobb-Douglas production function $\alpha(H) = \alpha(L)$, and therefore the risk-free asset is not traded in equilibrium. Second, notice that whether the position is positive or negative depends on the correlation between the income shares and output. If positive output shocks are associated with higher $\alpha(s)$, workers borrow on the risk-less asset and invest on the risky asset. By market clearing this in turn means that capitalists are issuing equity to increase their positive holdings of the risk-free asset to mitigate idiosyncratic risk. Third, regarding the risky asset, it is worth mentioning that if output is constant over time then $A^{\mathrm{CM}} = x^{\mathrm{CM}}$, so they hold the risky asset proportional to their relative level of wealth. If output is not constant, then workers and capitalists transfer consumption over time using the risky asset, holding positions that are more or less proportional to their wealth.

Now, we discuss the case with *incomplete markets*. As in Proposition 1, there are two reasons why asset positions change: changes in relative wealth and changes due to idiosyncratic risk due to $g^{CE}(\cdot)$. Note that the difference in positions, which we obtain from equations (18) to (21), is given by:

$$RB^w - R^{CM}B^{CM} = -\left(\frac{\alpha(H)-\alpha(L)}{\pi_2(H)-\pi_2(L)}\right)Y_2(L)Y_2(H)\left(x^{CM}-x_1\right) - \frac{\prod_s \alpha(s)Y_2(s)\left(g^{CE}(H)-g^{CE}(L)\right)}{\alpha(H)Y_2(H)-\alpha(L)Y_2(L)}x_1$$

$$A^w - A^{CM} = -\left(\frac{Y_2(H)-Y_2(L)}{\pi_2(H)-\pi_2(L)}\right)\left(x^{CM}-x_1\right) + \left[\left(g^{CE}(L)-1\right) + \frac{\alpha(H)\alpha(L)\left[g^{CE}(H)-g^{CE}(L)\right]}{\alpha(H)Y_2(H)-\alpha(L)Y_2(L)}\right]x_1$$

We now discuss the determinants of the *difference* in the allocations. It is easy to see that because $x^{CM} - x_1 > 0$ and $g^{CE}(H) - g^{CE}(L) > 0$, debt (principal plus interest) of workers is larger. As a result, capitalists, who take the other side of the trade, are increasing their position in the risk-free asset. Note that this result is magnified when the capital share changes more over states of nature, due to the term $\alpha(H) - \alpha(L)$. Moreover, it is interesting that the distortion to the holdings of the risky asset stems from three sources. The first source, captured by the term $g^{\mathrm{ce}}(L) < 1$, arises just because of the existence of uninsured idiosyncratic risk, and it remains even when α is constant. The second source, captured by the term $\frac{\alpha(H)\alpha(L)\left[g^{CE}(H)-g^{CE}(L)\right]}{\alpha(H)Y_2(H)-\alpha(L)Y_2(L)}$, arises because of the presence of "time-varying" uncertainty. Thus, the presence of uninsured idiosyncratic risk interacts with the stochastic income shares amplifying the difference between allocations. The third source, is due to the changing wealth shares that we just discussed.

The main take away from this section is that varying income shares open a wide range of new implications for financial markets, completely absent in theories that rely on the standard Cobb-Douglas and AK technologies.

2.5 Empirical predictions

When mapping the environment to the data, a natural question arises: Who are the capitalists issuing equity? How do we measure them? Since, in our setting, a single "capitalist" manages the firm and takes the decisions to insure her consumption, it may appear inconsistent to proxy them with corporations, which are the firms issuing equity to the public. Hence, capitalists would be better represented by private firms, whose owners have complete control over their businesses but do not issue equity.

There are two reasons why we think that our capitalist might be a good description of public corporations. First, we believe that a non-negligible fraction of corporations is not atomized, and there is an equity holder that bears id risk. A fitting example would be Tesla, which is a publicly-traded company, and as such it is owned by the public. So, one might be tempted to assume that the "owner-entrepreneur does not decide its fate." It is a corporation entirely owned by households (directly or indirectly as to all corporations), and as such, its governance would be more akin to that of a democratic organization. However, 17 percent of its shares still belong to Elon Musk, and he is still the leading "entrepreneur" in the company. That is the entrepreneur maximizing utility in our paper. Under the prism of our theory, giving away a large share of the ownership to the public is an optimal contract insuring against changes in the income shares. It is worth noting that Tesla is just an example, and we can find other as Amazon, Microsoft, Walmart, Apple in its origins, etc.

Second, we believe that this figure becomes larger if we count corporations in which the decision-maker bears idiosyncratic risk. What we feel is important to tilt the decision of the corporation to hedge id risk and do it more during good times (because the total variance is higher), is that the decision-makers bear id risk. In a recent interesting paper, Eisfeldt et al. (2021), find that equity-based compensation represents a large portion of the compensation of high-skilled workers. Note that a significant fraction of public firms have an atomistic equity base, but at the same time, top managers do bear id risk because their long-term compensation plan is equity-based. Our model also captures this ?deferred equity holder? which: holds little equity as a fraction of the total; a large fraction of their financial wealth is tied to the value of the firm; and they have a significant impact on the decisions.

Finally, note that a relevant prediction of our theory is that there should be a significant shift of companies legal structure from private business to corporate. We check this fact computing the ratio of the value of households' holding of nonfinancial private equity to nonfinancial corporate equity using the Flow of Funds tables. As we show in Appendix B.1, Figure 7 this ratio has significantly decreased from around 3 in 1980 to less than 0.5 in 2020. Thus, we take corporations as the unit of measure representing the firms managed by the capitalists and in what follow we will refer to them interchangeable with firms.

Regarding the empirical predictions, the implication that the accumulation of risk-free assets by corporations must be accompanied by households' increasing holdings of risky assets is an interesting and testable prediction of our theory that we fully address in the quantitative exercise. As a preview of our findings, we constructed, using the Flow of Funds for the U.S. economy, the holdings of direct and indirect equity by households (see Appendix B.1 for details). In Section 4.2, Figure 4 panel B, we show that there has indeed been a large increase in households' equity holdings, from 0.4GDP to 1.4GDP, an almost tripling of its value. In the same figure (panel A) we also plot all corporations' debt instruments that are unrelated to their main activity. We also see a steady increase of those, from around 0.07GDP to 0.21GDP. We select 1980 as the initial date in order to compare our results to other findings in related literatures that focus on a similar time frame. Nevertheless, in Appendix B.1 we show that the same pattern holds in a longer time span, which stresses that our mechanism is not just a salient feature of the last 30 years.

2.6 Normative implications

The allocation in the competitive equilibrium with incomplete markets features redistribution of wealth as a result of aggregate shocks. Is this redistribution a market failure? Can a planner alter the demand for insurance to improve the allocation? Turns out that no. In this section we show that the equilibrium in the two period model is constrained efficient. The notion of constrained efficiency follows Geanakoplos and Polemarchakis (1986) and Stiglitz (1982), and provides the planner with the same instruments as the market. In particular, the planner can intervene redistributing consumption across aggregate states with a lump sum transfer $T(s)$. Consumption for the consumer and the entrepreneur are given by:

$$c_2(s) = T(s) + (1-\alpha(s))Y_2(s)$$
$$e_2(s,i) = -T(s) + \alpha(s)g_iY_2(s).$$

Without loss of generality define $T(s) := \phi(s)Y_2(s)$. In Appendix E.2 we set up the planners problem, and we show that the first order conditions from the problem of maximizing the welfare of both consumers (given Pareto weights) subject to the technology constraints yields:

$$\frac{e_1^{-\gamma}}{c_1^{-\gamma}} = \frac{\mathbb{E}_i \left(-\phi(s)Y_2(s) + \alpha(s)g_iY_2(s)\right)^{-\gamma}}{\left(\phi(s)Y_2(s) + (1-\alpha(s))Y_2(s)\right)^{-\gamma}}.$$

Note that these are the same set of conditions as the ones in equation (9) for the Competitive Equilibrium, which implies that the Competitive Equilibrium is constrained efficient.[14]

3 General model

In this section we present the general model, allowing for any arbitrary $t \in \mathbb{N}$, any arbitrary number of aggregate states $s \in [s^1, s^2, ..., s^N]$ and an endogenous investment decision. To simplify notation, in what follows we characterize the solutions in a recursive fashion. In the two-period economy there was no investment, and given that after the second period there was no choice to be made, keeping track of the exogenous aggregate shock was enough. However, we also showed that the allocations depend on initial wealth distribution. In the infinite horizon economy the distribution of wealth will be changing along the business cycle. Thus, we will need to keep track of it, together with the effective stock of capital to determine the equilibrium. The redefined state space is $s = \{g_sK, x\}$, where x is the ratio of workers' wealth to total wealth. We formally show in Section 3.3 that these two state variables are enough to characterize the equilibrium. Since both K and x are endogenous, the transition function $\Pi(s'|s)$ is an equilibrium object. However, when solving the individual problems, in Subsection 3.1 and Subsection 3.2, the composition of s and how its transition is determined are irrelevant, because each individual takes them as exogenous.

[14]Things are different if the criteria for efficiency is Pareto Optimality. In the case of a Planner with enough instruments to perfectly control consumption in both aggregate and idiosyncratic states, it is worth noting that it will choose the same allocation as in the Complete Markets solution.

3.1 Workers

In this section we maintain the assumption that workers supply labor inelastically, but we extend the analysis to allow for Epstein-Zin preferences separating the Intertemporal Elasticity of Substitution (IES) from the risk aversion parameter. We do so building on Angeletos (2007).[15] Let σ be the inverse of the IES and γ the parameter governing risk aversion, then the worker solves:

$$\begin{aligned} V^{w}(a,s) \quad &= \quad \max_{\{c(s),a(s'|s)\}} \left\{ \frac{c(s)^{1-\sigma}}{1-\sigma} + \beta \left(\mathbb{E} V^{w} \left(a \left(s' \mid s \right), s' \right)^{\frac{1-\gamma}{1-\sigma}} \right)^{\frac{1-\sigma}{1-\gamma}} \right\} \\ st. \quad & \quad c(s) + \sum_{s'} p(s'|s) a(s'|s) \leq a(s) + \omega(s); \qquad \forall s, s' \end{aligned}$$

where $\omega(s)$ is the wage and $a(s'|s)$ are the AD securities bought by the consumer in state s, that pay off in the next period contingent on the realization of s'. The initial financial wealth $a_1 \equiv a(s_0)$ is given. To characterize the solution some definitions are in order. Denote by $h(s) = \sum_{s'|s} p(s'|s)[\omega(s') + h(s')]$ the worker's present value of future income (human wealth) and by $W^{w}(s) \equiv a + \omega(s) + h(s)$ her total wealth. In Appendix F we show that the solution is linear in total wealth, in the sense that both consumption and contingent asset's holdings can be expressed as linear functions of $W^{w}(s)$. They satisfy:

$$c(s) = (1 - \zeta(s)) W^{w}(s) \tag{22}$$

$$a(s'|s) = \phi^{w}(s'|s) \zeta(s) W^{w}(s) - \omega(s') - h(s') \tag{23}$$

where $\zeta(s)$ is the savings ratio out of total wealth and $\phi^{w}(s'|s)$ is the optimal wealth growth factor. This pair satisfies the following system of equations:

$$\phi^{w}(s'|s) = \left[\left(\frac{\zeta(s)}{(1-\zeta(s))} \right)^{\frac{1-\gamma}{1-\frac{1}{\sigma}}} \tilde{\beta}(s', s)^{\gamma} (1 - \zeta(s'))^{\frac{1-\gamma}{1-\frac{1}{\sigma}}} \right]^{\frac{1}{\gamma}}; \qquad \forall s, s' \tag{24}$$

$$\zeta(s)^{-1} = 1 + \left[\sum_{s'} p(s'|s) \tilde{\beta}(s', s) (1 - \zeta(s'))^{\frac{1-\gamma}{\gamma(1-\frac{1}{\sigma})}} \right]^{\frac{\gamma(1-\frac{1}{\sigma})}{1-\gamma}}; \qquad \forall s \tag{25}$$

[15] Allowing for an endogenous labor supply would not affect the bulk of our analysis. It would certainly change our calibration but, given that we are targeting the level and fluctuations of the income shares, the conclusions would remain valid as long as they are driven by technology.

where $\tilde{\beta}(s'|s)$ is given by:

$$\tilde{\beta}(s'|s) = \frac{\beta^{\frac{(1-\gamma)}{\gamma(1-\sigma)}} \Pi(s'|s)^{\frac{1}{\gamma}}}{p(s'|s)^{\frac{1}{\gamma}}}; \qquad \forall s, s'. \tag{26}$$

It is clear that taking prices and probabilities $p(s'|s)$ and $\Pi(s'|s)$ as given, equation (25) is a fixed point in $\zeta(s)$. Once we find $\zeta(s)$, equation (24) solves for the state contingent assets holdings.[16]

3.2 Capitalists

Technology. Capitalists combine labor and capital to produce using a constant returns to scale technology:

$$y(g_s, g_i, k, l) = F(g_i g_s k, l) + (1-\delta) g_i g_s k$$

where k is stock of capital, l is labor, δ is the depreciation rate and g_i, g_s represent the idiosyncratic and the aggregate shocks, respectively. Both are assumed to be *i.i.d* over time. Denote by $k(s,i) = g_i g_s k$ the *effective capital stock*. The firm hires labor in competitive markets. In Appendix C.1 we show that the income from capital, $F(g_i g_s k, l) + (1-\delta) g_i g_s k - \omega(s) l(s)$ can be written as:

$$\pi(s,i) = g_i R(s) k(s) \tag{27}$$

where $R(s) = (1-\delta) g_s + r(s)$, with $r(s) = \frac{\partial y(s, \mathbb{E} g_i, k, l)}{\partial k}$. Since we assume that $E(g_i) = 1$, the aggregate capital income share is affected only by the aggregate shock. Notice the linearity of profits in the capital stock and the idiosyncratic shock. This is instrumental in the characterization of the equilibrium as it allows for linear decision functions.

Contracting. Capitalists are subject to idiosyncratic risk, so they will try to insure it. Following DeMarzo and Fishman (2007) we assume that they have access to risk neutral intermediaries who can provide insurance. However, due to moral hazard, there is a limit on how much idiosyncratic risk can be offloaded. To be precise, we model moral hazard as endowing capitalists with the possibility of diverting resources from the firm to their private accounts at a cost $0 < 1-\psi < 1$. For each unit of profit that they divert, only ψ units are transformed into consumption goods (or savings). The contract stipulates that the capitalist must hand over to the financial intermediary a given proportion of her risky profits, receiving an average of the profits of all firms in return. Since capitalists

[16] It is straightforward to show that in the special case in which $\gamma = \sigma$, CRRA utility, the system is linear in $(1-\zeta(s))^{-1}$, greatly simplifying the solution. In a previous draft we show how to solve this setting by simple matrix inversions.

can misreport their profits and consume a proportion ψ of the misreported profits, in Appendix E we show that the optimal contract implies that she must retain (or be exposed to) a proportion ψ of the idiosyncratic risk. This is known in the literature as a "skin in the game" constraint.[17] As a result, we can write the exposure to the idiosyncratic risk in a simple reduced form. Let $\tilde{g}_i \geq 0$ be the productivity shock to which the firm is exposed. Then, an economy with idiosyncratic risk $\tilde{g}_i$ and restricted insurance is equivalent to an alternative economy in which individual risk is not insurable and firms are subject to idiosyncratic risk g_i satisfying:

$$g_i = (1-\psi)\mathbb{E}_i\tilde{g}_i + \psi\tilde{g}_i \geq 0. \tag{28}$$

Program. Consistent with the worker's problem we also assume that capitalists are endowed with Epstein-Zin preferences and the same parameters as workers.

$$\begin{aligned} V^e(E,k;s,i) = &\max_{\{e(s,i),E(s'|s),k'(s,i)\}} \left\{ \frac{e(s,i)^{1-\sigma}}{1-\sigma} + \beta \left(\mathbb{E}V^e\left(E',k',s',i'\right)^{\frac{1-\gamma}{1-\sigma}} \right)^{\frac{1-\sigma}{1-\gamma}} \right\} \\ &s.t. \quad e(s,i) + k'(s,i) + \sum_{s'} p(s'|s)E(s'|s) \leq E(s) + g_i R(s)k; \qquad \forall i,s,s' \end{aligned}$$

where $E(s'|s)$ are AD securities bought by the capitalist in state s, with payoffs contingent on the realization of state s'. The initial financial wealth $E_1 \equiv E(s_0)$ is given. In this section we show that, despite being subject to idiosyncratic risk, the consumption and savings ratios are simple and akin to those of the workers. Due to homothetic preferences, savings ratios are linear in wealth, and thus total savings are independent of the distribution of wealth. There is aggregation: knowing the average net worth is enough to forecast future aggregate capital. For this result it is crucial that individual returns are a linear function of the individual holdings of capital as shown in Appendix C.1.

In Appendix F we show that analogously to the workers problem, the capitalist's optimal choices are linear in wealth. Her total wealth is $W^e(s,i,k) = E(s,i) + R(s)g_ik$ (recall that $R(s)$ is a gross rate), which allows us to write the optimal decisions as:

[17] DeMarzo and Fishman (2007) assume that the principal can sign long-term contracts (there is commitment) and that both the principal and the agent are risk neutral. In contrast, we consider a risk averse agent who can only commit to short term contracts. For similar setups and results in continuous time see DeMarzo and Sannikov (2006). We also show that as long as insurance contracts are not history dependent, this is the best possible insurance independently of whether or not the entrepreneurs have access to hidden savings. This contract is akin to an equity contract in which the entrepreneur creates a company, issues equity for a proportion $1-\psi$ of its ex-ante value and retains a proportion ψ of the value of the company. See Di Tella (2019) for an example of how a social planner could improve the allocations using taxes.

$$e(s,i) = (1-\vartheta(s))W^e(s,i,k) \tag{29}$$

$$k'(s,i) = \nu(s)\vartheta(s)W^e(s,i,k) \tag{30}$$

$$E(s'|s,i) = \phi^e(s'|s)\vartheta(s)(1-\nu(s))W^e(s,i,k) \tag{31}$$

where $\vartheta(s)$ is the entrepreneur's savings ratio, $\phi^e(s'|s)$ the optimal financial wealth growth factor, and $\nu(s)$ is the portion of savings invested in capital. In what follows we will refer to $\nu(s)$ as the investment rate. Note that the law of motion of individual wealth is:

$$W^e(s',i',k') = \vartheta(s)o(s',i;\phi^e,\nu)W^e(s,i,k) \tag{32}$$

where

$$o(s',i';\phi^e,\nu) \equiv (1-\nu(s))\phi^e(s'|s) + \nu(s)R(s')g_{i'}$$

Again, the solution of the problem can be characterized by a system of equations solving for $\vartheta(s)$, $\phi^e(s'|s)$ and $\nu(s)$ which is given by:

$$\vartheta(s)^{-1} = 1 + \left[\sum_{s'} p(s'|s)\tilde{\beta}(s',s)(1-\vartheta(s'))^{\frac{1-\gamma}{\gamma(1-\frac{1}{\sigma})}} \frac{\mathbb{R}(s',s)^{\frac{1}{\gamma}}}{1+Prem(s)}\right]^{\frac{\gamma(1-\frac{1}{\sigma})}{1-\gamma}} \tag{33}$$

$$o(s',i';\phi^e,\nu) = \left(\frac{\vartheta(s)(1-\vartheta(s'))}{(1-\vartheta(s))}\right)^{\frac{1-\gamma}{\gamma(1-\frac{1}{\sigma})}} \tilde{\beta}(s',s)\mathbb{R}(s',s)^{\frac{1}{\gamma}} \tag{34}$$

$$\mathbb{E}_{s',i'|s}\left[(1-\vartheta(s'))^{\frac{1-\gamma}{1-1/\sigma}}\left[o(s',i';\phi^e,\nu)\right]^{-\gamma}\left(R(s')g_{i'} - \frac{1}{\sum_{s'|s} p(s'|s)}\right)\right] = 0 \tag{35}$$

where $Prem(s) \geq 0$ is the risk premium and $\mathbb{R}(s',s) \geq 1$ is a risk adjustment factor shown in the appendix, so that when $\mathbb{V}ar(g_i) = 0$ then $Prem(s) = 0$ and $\mathbb{R}(s',s) = 1$.[18]

Some features about the system are worth noting. First, all three objects are independent of individual wealth and the current idiosyncratic shock. This independence of the equilibrium from the underlying distributions is due to the linearity of the decision functions and greatly simplifies the analysis. Second, note that the main difference between (25) and (33) is $\frac{\mathbb{R}(s',s)^{\frac{1}{\gamma}}}{1+Prem(s)}$. When there is no idiosyncratic risk this term is equal to 1 so capitalists choose the same savings ratios as workers. However, in general the term is

[18]See equation (72) in Appendix F, where we provide a definition, making explicit the dependency of it on both $\mathbb{V}ar(g_i)$ and $\phi^e(s',s)$.

bigger than one. As a result, in equilibrium, for any price function $p(s)$, it must be true that $\vartheta(s) > \zeta(s)$: on average capitalists save more than workers. This creates a downward drift on the worker's wealth ratio, x. As we showed in the two-period economy, this wealth effect has important quantitative implications, generating large changes in the position of financial assets.[19]

Finally, equation (34) pins down the wealth growth factor. Comparing (24) and (34) we see that the capitalist's savings ratio is affected by the additional term $\mathbb{R}(s', s)^{\frac{1}{\gamma}}$, which depends on both risk aversion and the exposure to uninsured idiosyncratic risk. In the absence of uninsured idiosyncratic risk both agents would react equally to aggregate shocks.

3.3 Equilibrium

The allocations must satisfy the assets' and goods' market clearing conditions, which pins down the equilibrium prices $p(s'|s)$. Furthermore, $\Pi(s'|s)$ must be consistent with the laws of motion generated by individual decisions. The assets' and goods' market clearing conditions are:

$$a(s'|s) + E(s'|s) = 0 \qquad \forall s, s' \tag{36}$$

$$c(s) + e(s) + K'(s) = y(s) \quad \forall s \tag{37}$$

where $e(s) = \int_i e(s, i, k, E)$, $K'(s) = \int_i k'(s, i, k, E)$, $y(s) = \int_i y(s, i, k, E)$ and $E(s'|s) = \int_i E(s'|s, i, k, E)$. We have avoided the dependency of the allocations on individual wealth because the savings and investment ratios are independent of it.

Let the total wealth be $W^T(s) = W^w(s) + W^e(s)$ and define the worker's wealth ratio as $x = W^w(s)/W^T(s)$. In Appendix F, equation (83), we show that the AD prices satisfy:

$$\frac{p(s'|s)}{\beta^{\frac{(1-\gamma)}{(1-\sigma)}}\Pi(s'|s)} = \left[\left(\frac{\zeta(s)(1-\zeta(s'))}{(1-\zeta(s))}\right)^{\frac{1-\gamma}{\gamma(1-1/\sigma)}}\zeta(s)x + \left(\frac{\vartheta(s)(1-\vartheta(s'))}{(1-\vartheta(s))}\right)^{\frac{1-\gamma}{\gamma(1-1/\sigma)}}\vartheta(s)\mathbb{R}(s', s)^{1/\gamma}(1-x)\right]^{\gamma}\left(\frac{W^T(s)}{W^T(s')}\right)^{\gamma} \tag{38}$$

All of the elements in this equation are endogenous, which complicates the interpretation. However, in the following sections we show how this equation changes under different assumptions, clarifying the economic mechanisms at play.

[19] The drift also implies that in the limit the workers end up with zero wealth. This may seem like an odd prediction, but it is the natural outcome of combining agents with heterogeneous exposure to risk. To construct equilibria with non-degenerate wealth distributions, the literature has resorted to alternative strategies. One solution is to introduce different β's, with capitalists discounting the future more, as Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2012). Alternatively, one can assume that with some exogenous probability capitalists and workers switch "functions", while maintaining their wealth. In the next section, for our quantitative exercise, we follow the first approach.

It is simple to show that the state variables satisfy the following laws of motion:

$$K'(s) = \nu(s)\vartheta(s)(1-x)W^T(s) \tag{39}$$

$$x(s'|s) = \phi^w(s'|s)\zeta(s)\frac{W^T(s)}{W^T(s')}x \tag{40}$$

Notice that both are Markovian, so it is possible to compute their transition probabilities. As result, (39) and (40) together with the exogenous probability distribution over g_s determine the transition probabilities $\Pi(s'|s)$.

3.4 Benchmark economies

In this section, Proposition 3, we show that the insights presented in Proposition 1 extend to the infinite horizon economy with endogenous investment. These results will be useful to understanding the quantitative implications that we discuss in Section 4. For simplicity we consider the case in which $\sigma = \gamma$, but our results extend to the general case of EZ preferences. Define $\tilde{g}(s', s) := \frac{Y(s')}{Y(s)}$.

Proposition 3. *Suppose that $\sigma = \gamma$. Then:*

(a) If $\mathbb{V}ar(g_i) = 0$, then $x(s'|s) = x$ for all (s, s').

(b) If $\mathbb{V}ar(\alpha(s)) = 0$, and output growth is state independent, then there exists a $\beta^e < \beta$: (i) $x(s'|s) = x$ for all (s, s') and (ii) $p(s'|s) = \beta\Pi(s'|s)\tilde{g}(s', s)^{-\gamma}$ for all (s, s').

Proof. See Appendix H. □

Complete Markets: $\mathbb{V}ar(g_i) = 0$**.** Part (a) of Proposition 3 extends, to a more general environment, the result in part (b) of Proposition 1. After any aggregate shock s, and after any history of shocks, the portfolio holdings are such that all the implied payments leave both agents with the same *relative* wealth. In Proposition 1 we were able to obtain a sharp characterization of the evolution of wealth shares because the second period wealth was equal to income. Thus, the compensation consisted of only the difference between the income share and the wealth ratio x. In the general setup the compensation embeds not only the current difference in income but also the present value of all the expected future changes.

The analogy is also evident in the implied asset prices. Using equation (38), because with complete markets $\zeta(s) = \vartheta(s)$ and $\mathbb{R}(s', s) = 1$, the AD prices are given by:

$$p(s'|s) = \beta\Pi(s'|s)\left(\frac{1-\vartheta(s)}{1-\vartheta(s')}\right)^{\gamma}\left(\frac{W^T(s)}{W^T(s')}\right)^{\gamma}$$

The differences with respect to Proposition 1 are: 1) the price depends on the ratio of wealth rather than the ratio of second period output, and 2) the ratio of the marginal propensity to consume also appears. This is because in Proposition 1 there was no investment decision, while here the resources diverted to investment change over the business cycle. If the consumption ratios were constant, the prices would just reflect the random growth in total wealth. This happens because the changes in $W^T(s)$ reflect the common component of the shock, and therefore cannot be insured.

Constant Income Shares: $\mathbb{V}ar(\alpha(s)) = 0$. In the case in which the capitalist and the workers share the same discount factor, because capitalists are exposed to idiosyncratic risk, they tend to accumulate more assets than workers, due to a precautionary savings motive. In the limit, this force pushes the proportion of wealth in workers' hands towards zero. To compensate for this downward drift one can assign a smaller discount factor β^e to capitalists.[20]. This alternative discount factor satisfies:

$$\beta^e = \beta\frac{(1+D)^{-\sigma}}{\mathbb{E}_i(1+Dg_i)^{-\sigma}}$$

where $D > -1$ is the ratio of risky physical investment to financial assets in the capitalist's portfolio, capturing the capitalist's exposure to idiosyncratic risk. Only when $\mathbb{V}ar(g_i) = 0$ then $\beta^e = \beta$, while for any strictly positive variance $\beta^e < \beta$. With this adjustment, Proposition 3 generates the same prices and allocations as an analogous economy with complete markets ($\mathbb{V}ar(g_i) = 0$) and $\beta = \beta^e$. As we discussed in Section 2.6, *fluctuations in the income shares are key for generating non-degenerate financial portfolios which interact with the shocks amplifying its effects through changes in the quantities*.

Similar results can be found in the literature. Proposition 3 is a generalization of Di Tella (2017), which presents a similar result in a continuous time environment with an AK technology (and hence no labor supply) and aggregate productivity shocks that follow a Geometric Brownian Motion. We extend the result to a discrete time environment, allowing for any CRS technology and an additional factor of production (labor), as long as the income shares are constant. Also Bocola and Lorenzoni (2020) provide a similar result in a discrete time environment, but they maintain the AK assumption and assume that the aggregate productivity shock is *i.i.d.* (in levels) over time.

[20]This result is known since Blanchard (1985) and Yaari (1965). More recent discussions are Gârleanu and Panageas (2015) and Di Tella (2017)

3.5 The economy with moving shares

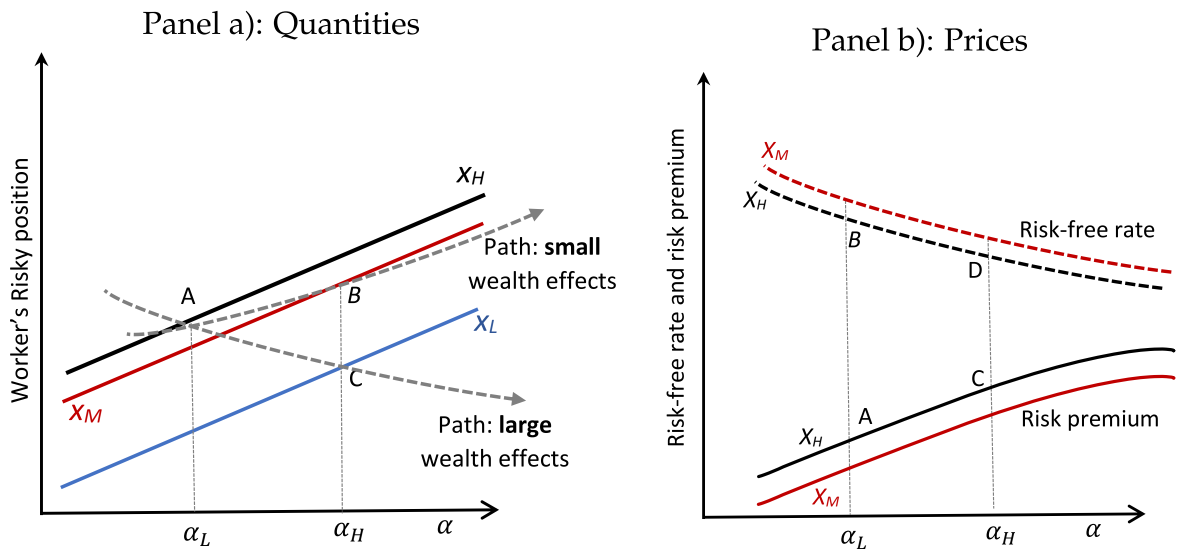
Obtaining analytical results with varying shares and idiosyncratic risk is less straightforward in the general model. However, using the insights from Proposition 1 and the equilibrium equations we can provide some intuition for the expected outcomes.

We start by analyzing the effects of an increase in the capital share on the desired financial positions. The assets' positions widen with α, so that a larger α leads workers to a larger positive position in the risky asset, leveraging in the risk-free asset. For instance, suppose that the economy is initially in a state with capital share α_L and distribution of wealth x_H. In Panel a) of Figure 1, we depict the hypothetical worker's risky position at point A. Now suppose there is a shock that increases the capital share to $\alpha_H > \alpha_L$. Absent wealth effects, the worker's position would move along the black line labeled x_H. In this case, a positive α shock unambiguously increases the worker's position on the risky asset. The same argument applies for the worker's debt. This would be the outcome with complete markets, since x would remain constant after any shock.

However, when markets are incomplete, a positive α shock would increase the capitalist's wealth, and therefore x would fall. Since a larger $1 - x$ implies that more resources must be devoted to insure the idiosyncratic risk, capitalists are less willing to trade on the insurance of the income shares' risk. This effect dampens the increase in the trading of the risky asset; it does so in such a way that after the shock the worker's risky position may end up being smaller. If the wealth effect is "small", say x moves to a new level $x_M < x_H$, such that the demand now lies on the red line labeled x_M, then the new position would be located at a point such as B in Panel a) of Figure 1. Despite the dampening effect, the worker's risky positions would be positively correlated with the capital share. But if the wealth effect is large enough, e.g. the wealth distribution moves to $x_L < x_M < x_H$, then the economy would end up at point C, where the worker's risky position and α are negatively correlated. Thus, the extent of uninsured idiosyncratic risk and its implications for the wealth effects are crucial components of the quantitative implications.

In Panel b) of Figure 1 we show the expected patterns for asset prices. To this end it is important to bear in mind equation (38). In the previous section neither idiosyncratic risk nor the distribution of wealth played any role, but now these two components matter. In our setup the equivalent to a risk-free rate is given by $1/\sum_{s'} p(s'|s)$. An increase in the "average" price is equivalent to a fall in the risk-free rate. As in the two-period model, an increase in α acts as an increase in uncertainty, which is reflected in a larger factor $\mathbb{R}(s', s)$ in equation (38). This direct impact is the main component generating a decreasing risk-free rate as shown in Panel b). Ceteris paribus, the new rate would move from point B to point D. However, there are two additional effects. First, x drops, say to $x_M < x_H$. Then

Figure 1: Wealth effects and financial markets



the weight on $\mathbb{R}(s', s)$ increases, which also raises the average prices. However, because of the increased risk, the capitalists' consumption slows down, which puts a downwards pressure on prices. Taken together, these simultaneous effects could dampen the fall in the risk-free rate, as shown in Panel b) with the line labeled x_M, or reinforce it.

Moreover, in Appendix F (see equation (76)) we show that the risk premium satisfies:

$$Prem(s) = \sum_{s'|s} p(s', s) \left[\frac{\sigma v(s)^2 r(s')^2}{o(s', 1)\mathbb{R}(s', s)} \mathbb{V}ar(g_i) \right] \tag{41}$$

In this case the larger α has a direct and sizeable impact on increasing the risk premium. Without wealth effects the risk premium should increase as depicted in the shift from A to C in Panel b). But there are two additional indirect effects related to the wealth distribution. First, because the wealth share of workers drops, who are not exposed to idiosyncratic risk, it becomes increasingly harder to insure it, and therefore the risk premium could rise further. Second, the higher exposure to the idiosyncratic risk generates a portfolio rebalancing, in which the capitalist invests less in capital so that $v(s)$ falls. Thus, the risk premium may fall as indicated in Figure 1. Which effect dominates is a quantitative question that we analyze in the next section.

4 Quantitative implications

We now present the quantitative results. In a nutshell, we showed that income shares' risk alone generates non-trivial portfolio allocations, but due to the lack of wealth effects, the allocation is invariant to the state of the economy. Uninsured idiosyncratic risk alone delivers relevant wealth effects, but the allocations are still invariant to aggregate shocks and imply degenerate portfolios. In this section we quantitative solve the model and we show how relevant are the effects discussed in the previous section.

It is important to stress that we do not attempt to match either the assets' positions or prices of those assets. There are multiple factors affecting the financial markets from which we are abstracting. For instance, households may want to accumulate risk-free assets for liquidity reasons, but we have not included a demand for liquidity. For this reason, we calibrate the model to replicate "untargeted" standard moments and we evaluate its predictions for the financial markets.

4.1 Calibration

To construct the mapping to the intuitive risk-free and risky asset positions we assume that the aggregate shock can take on two values: g_H and g_L, each occurring with probability $1/2$. The *i.i.d.* structure of the shock simplifies the state space without losing realism, since due to the permanent effect on the capital stock the generated output is close to a random walk. In addition, with only two possible realizations of s we can construct the straightforward mapping from the economy with AD securities to that with only two assets: a risk-free and a risky asset. Adding more realizations would have minimal quantitative effects and would make this mapping less clear.

To discipline the relationship between output and the capital share, with meaningful variation, we use a CES production function with parameters $\{\rho, \alpha_k\}$:

$$F(K,L) = \left[\alpha_k K^{\frac{\rho-1}{\rho}} + (1-\alpha_k) L^{\frac{\rho-1}{\rho}}\right]^{\frac{\rho}{\rho-1}}$$

Thus, two key parameters are α_k, which determines the average capital income share, and ρ which pins down the correlation between output and the capital share. As a result, we need to calibrate 10 parameters, which in turn we group in two sets: 1) those borrowed from previous studies, $\{\beta, \sigma, \gamma, \psi, \mathbb{V}ar(g_s)\}$ and, 2) those chosen to replicate aggregate moments $\{\beta^e, \delta, \alpha_k, \rho, \mathbb{V}ar(g_i)\}$. We now discuss them sequentially.

Regarding the parameters borrowed from the literature, we set the worker's discount factor to $\beta = 0.96$ and the inverse of the IES to $\sigma = 2$. As shown by Crump et al. (2015) most empirical studies point towards an IES of 0.5.[21] These are both standard parameters in the literature. Since the worker discounts the future less, it would be the agent determining the average risk-free rate. Thus, our choice implies a risk-free rate of around 1% in the stationary equilibrium. Also, following the literature we set $\gamma = 5$ which together with the other sources of risk determine the risk premium.

Two important determinants of the risk premium are ψ and $\mathbb{V}ar(g_s)$. There is ample evidence that firms are more exposed to idiosyncratic risk than workers. Following He and Krishnamurthy (2012), we set $\psi = 0.2$ to match the 20% share of profits that hedge funds charge. Moreover, since the aggregate shock can take on only two values we set $g_H = 1.02$ and $g_L = 0.98$. Notice that the variance of the assumed process is $\mathbb{V}ar(g_s) = p(1-p)(g_H - g_L)^2 = \frac{1}{4}0.04^2 = 0.0004$, which is in line with the medium-long term variation of GDP in the U.S. economy.

The remaining parameters are set to replicate aggregate moments for the U.S. economy. We chose β^e to obtain the implied average x in the data, where x is computed using the Flow of Funds tables. Since $x = \frac{W^w}{W^T} = \frac{A/y+(1-\alpha)+h/y}{1+(1-\delta)k/y+h/y}$, we use households' financial assets over GDP as a measure of A/y. We approximate h/y as $h/y = (1+E(r))E(1-\alpha)/E(r)$, which is the exact value for the human wealth to GDP ratio in a deterministic economy. For the risk-free rate we use the Fred AAA 10 year corporate bond. Then using the capital to GDP ratio from Fred in every period we obtain that the average x is around 0.82. This approach generates $\beta^e = 0.8725$.

There are many reasons why the labor share could be procyclical in the medium-long run, which are not yet fully understood. It could be due to purely technological reasons, as in Karabarbounis and Neiman (2014) and Koh et al. (2020), or changes in bargaining power during the business cycle, as in Ríos-Rull and Santaeulàlia-Llopis (2010), or just wage rigidities as in Cantore et al. (2019).[22] Since our model does not include additional frictions, we rely on ρ to obtain income shares variation.[23] Moreover, with this technology the capital income share also depends on the capital-output ratio. Thus, we set $\rho = 1.5$ and we choose α_k and δ to jointly target the average capital-output ratio and the average

[21]This would affect the resulting equity premium. A large literature has explored alternative solution such as long-run risk, Bansal and Yaron (2004) and Hansen et al. (2008), and disaster risk, Barro (2009) and Gourio (2012), as possible explanations for the premium between equities and safe bonds.

[22]Karabarbounis and Neiman (2014) and Koh et al. (2020) find values of ρ between 1.15 and 1.4; Cantore et al. (2019) and Ríos-Rull and Santaeulàlia-Llopis (2010) find that the labor share is pro-cyclical in the medium-long run. León-Ledesma and Satchi (2018) show how to model an economy in which the ES is different in the short run than in the medium-long run based only on technology choices.

[23]The variance of the capital share in the data ranges from 1.4% to 2%, depending on how the labor share is computed. To obtain this value we need to force ρ to reach unrealistically high values. For instance, Karabarbounis and Neiman (2014) finds that ρ could range from 1.15 to 1.4.

Table 1: Baseline Calibration

Parameter	Description.	Value
γ	Risk aversion.	5
σ	IES inverse.	2
β	Workers' discount factor	0.96
β^e	Capitalists' discount factor	0.8725
ρ	Elasticity of Substitution	1.50
α_k	Capital Share Parameter	0.265
δ	Depreciation	0.075
$g_{s,h}, g_{s,l}$	Aggregate Shocks to Capital	1.02, 0.98
p_s	Probability of g_s	0.5
$Var(g_i)$	Variance of Id. Shocks to Capital	0.0182
ψ	Exposure to Id. Risk	0.20

capital share. This generates $\alpha_k = 0.265$ and $\delta = 0.075$. The resulting average capital share is 0.37, in line with most estimates that include sufficiently long time series.[24] We obtain a capital-output ratio of around 2.78, which is in line with most estimates of roughly 2.7.

Finally, given the risk aversion parameter and aggregate volatility, the risk premium is determined by the exposure to idiosyncratic risk. That is, what matters for capitalists is the residual risk $\psi^2 \mathbb{V}ar(g_i)$. Since we are assuming that workers are not subject to idiosyncratic risk, this risk must be interpreted in relative terms. To this end we target a risk premium of 6%, which generates $\mathbb{V}ar(g_i) = 0.04$ and thus the total idiosyncratic risk borne by entrepreneurs is given by $\psi^2 \mathbb{V}ar(g_i) = 0.728 \times 10^{-3}$. We summarize the calibrated parameters in Table 1.

4.2 Results: the corporate and household gluts

Table 2 displays several moments of the calibrated economy. Panel A shows the targeted values in the data and the corresponding model predictions. On average, the capital share is around 0.37, the capital-output ratio is 2.78 and the workers' share of wealth is 0.82, very close to the targeted moments. Panel B displays the values for the risk-free rate, the investment rate and the workers' portfolio allocations. Although these are all non-targeted statistics, the calibrated economy delivers sensible predictions for each quantity. The risk-free rate is on average 1%, investment is 21% of GDP and, on average,

[24]The current debate about the labor share focuses on the last 20 years or so and estimates lower values. In our environment the capital share is very close to a random walk but is still stationary. As we show in the next section, long periods of time (say 20 to 30 years) with either constantly increasing or decreasing capital shares arise naturally.

Table 2: Simulated moments

Quantity	Description	Data	Model
Panel A - Targeted Moments - Means			
$K(s)/Y(s)$	Capital-output ratio	2.50	2.78
$\alpha(s)$	Capital income share	0.37	0.37
$x(s)$	Workers' wealth share	0.82	0.82
Panel B - Non-targeted Moments - Means			
$B(s)/Y(s)$	Worker risk-less asset position		-1.60
$PA(s)A(s)/Y(s)$	Worker risky asset position		0.80
$r(s)$	Risk-free rate		0.01
$I(s)/Y(s)$	Investment		0.21
$Prem(s)$	Risk premium		0.05

the workers hold a positive amount of risky assets (equities) and finance this position by borrowing on the risk-less asset. The assets' positions have been constructed using the equilibrium laws of motion in a similar fashion to Section 2.

The main takeaway from Table 2 is that the mechanism of this paper is able to generate large and reasonable financial positions with apparently low variations in the income shares. Situating the values in the context of the U.S. economy, households held around 0.8GDP in equity in 2018, while the total private debt was also around 1.6GDP. Moreover, the model delivers sensible results for financial positions while also implying an underlying risk-free rate and risk premium that are not far from the actual observations. Last but not least, the asset positions and prices exhibit the right sign for the correlations with both GDP and the capital share.

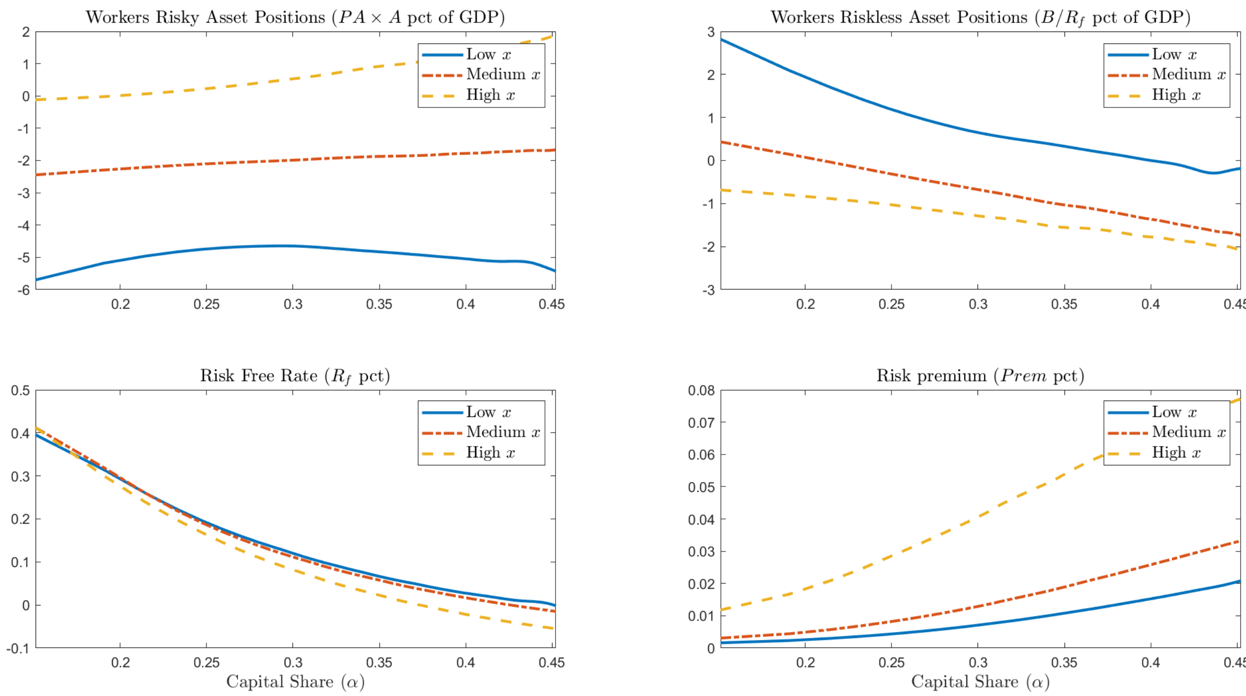
Our main result is illustrated in the top half of Figure 2, which shows the calibrated version of Figure 1, Panel a). In the top left panel of Figure 2 we plot the risky asset positions, $PA(s) \times A(s)/Y(s)$, for different values of x. As discussed in Section 3.5, the position on the risky asset widens as $\alpha(s)$ increases and the wealth effects become more prominent. For instance, when x falls from 0.84 to 0.58, the risky position decreases by around 3 GDP (the difference between the yellow dashed line and the red dotted line). In fact, if x is sufficiently low, the worker will also be borrowing in the risky asset. The flip side of the risky assets accumulation is the borrowing in the risk-free asset. In the top right panel of Figure 2 we depict the implied patterns for the workers' holdings of risk-free assets. The pattern for $A(s)$ is mirrored by $B(s)$ with the opposite sign. As the capital share increases, the financial positions widen, increasing the leverage that is used to accumulate risky assets. Because of market clearing, the capitalists' financial positions are the negative of workers' positions. Thus, if workers are borrowing to buy equity, it must be that capitalists is increasing the issuance of equity and accumulating risk-free assets.

The increased holdings of risk-free assets by corporations is known as the *corporate saving glut*. What we add to this discussion is the implication that it must be accompanied by increasing holdings of risky assets by households: a *household's equity glut*. Given the discussion in Section 3.5 and the large wealth effects observed in Figure 2, one may wonder if the time paths implied by the model generate a positive or negative correlation between the capital share and the capitalists' holdings of risk-free assets. To shed light on this issue we reproduce an increasing path for α of a similar magnitude to the observed in the data since 1980. We plot the implied paths for the worker's wealth share, risk-free assets, risky assets, risk-free rate and the risk premium. The results can be seen in Figure 3. With a standard calibration, the wealth effects are not enough to overturn the patterns predicted by the complete markets economy. As the labor share falls, capitalists accumulate more risk-free assets, and lend these funds to workers who, leveraging, invest in equity to insure against changes in the income shares. At the same time, as the capital share increases, it becomes harder for capitalists to insure the idiosyncratic risk, which put a downwards pressure on the risk-free rate. Because capital is a risky asset, to decrease their exposure to risk capitalists also invest less.

The bottom half of Figure 2 illustrates the asset prices implications of the model economy. This is the calibrated version of Figure 1, Panel b). First, in the lower left panel is evident that the risk-free rate sharply falls as the capital share increases and that the wealth effect is mild. The combination of high exposure to idiosyncratic risk and large quantities of accumulated capital pushes the return on capital downwards. As expected from Section 3.5, the risk premium increases as the capital share rises as depicted in the bottom right panel of Figure 2. Unlike the risk-free rate, here the wealth effects are more relevant. For a given capital share, the risk premium is higher the higher is the consumer's wealth share.

Two issues are worth discussing. First, notice that wide variation on the risk-free rate, ranging from 40% for very low capital shares, to negative rates for capital shares above 0.4. Although it may appear puzzling and exaggerated, this is a natural implication of the CES production function together with the arbitrage condition between assets and it is consistent with the data. Because $\rho > 1$, low capital shares are only consistent with very low levels of capital. Thus, in that region, due to the decreasing marginal productivity, the return on capital is large, to the point that as $\lim K \to 0$ the marginal productivity of capital approaches infinity. Then, by arbitrage, the risk-free rate must also be large. Still,

Figure 2: Equilibrium policy functions



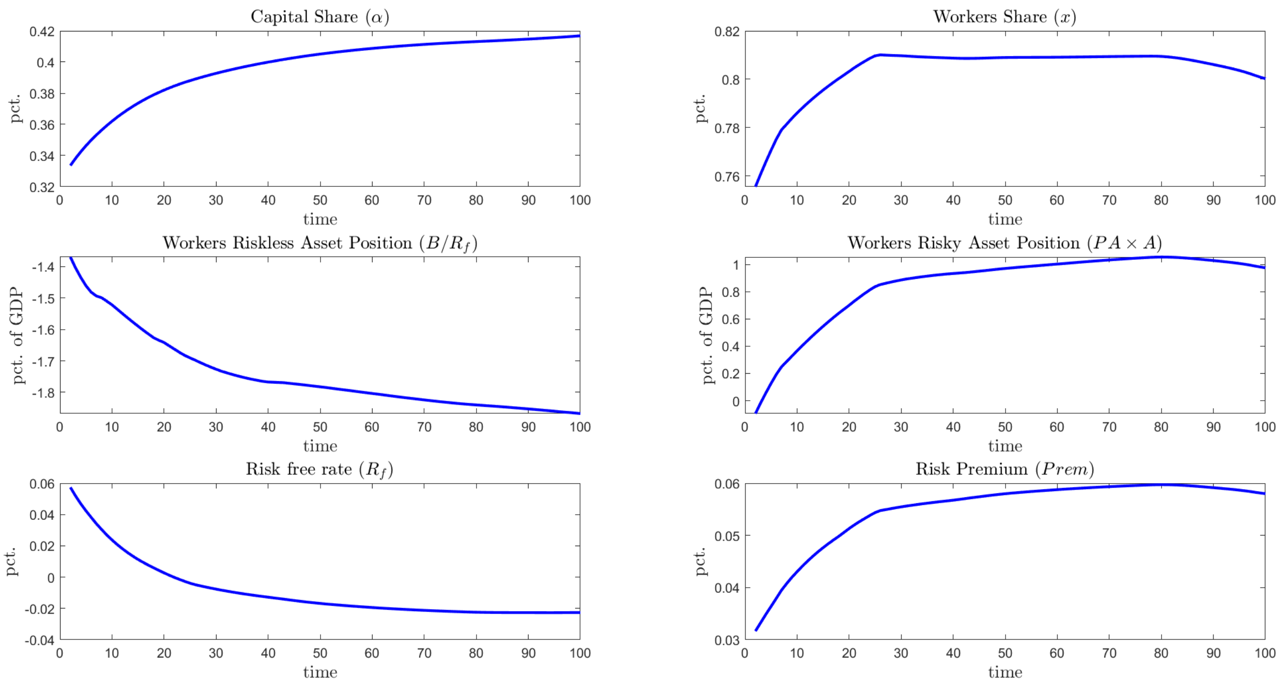
Note: This figure plots, from left to right and from top to bottom, the risk free rate, the risk premium and the risky and risk-less asset positions for consumers as a function of the capital share. The values of x low, medium and high are given by 0.31, 0.58, 0.84.

for empirically relevant levels of the capital share, the risk-free rate remains around the expected levels.[25]

Second, notice that the risk premium sharply reacts to changes in the capital share and the wealth ratio. The steep upward sloping shape is due the "direct" effect generated by a larger α. As discuss in Section 2.3's two period model (equation (12)) and in Section 3.5's general characterization (equation (75)), an increase in α is akin to an increase in the supply of risk. Since this increase comes at the expense of a lower absorbtion capacity by agents not exposed to risk (workers), the risk premium increases. This effect tends to generate a positive correlation between the risk premium and the capital share and, therefore, with GDP when $\rho > 1$. Moreover, keeping α constant, the risk premium is also increasing on the worker's wealth share. This happens due to "indirect" wealth effects. A lower capitalist's wealth share, on the (capital) investment rate requires a larger compensation

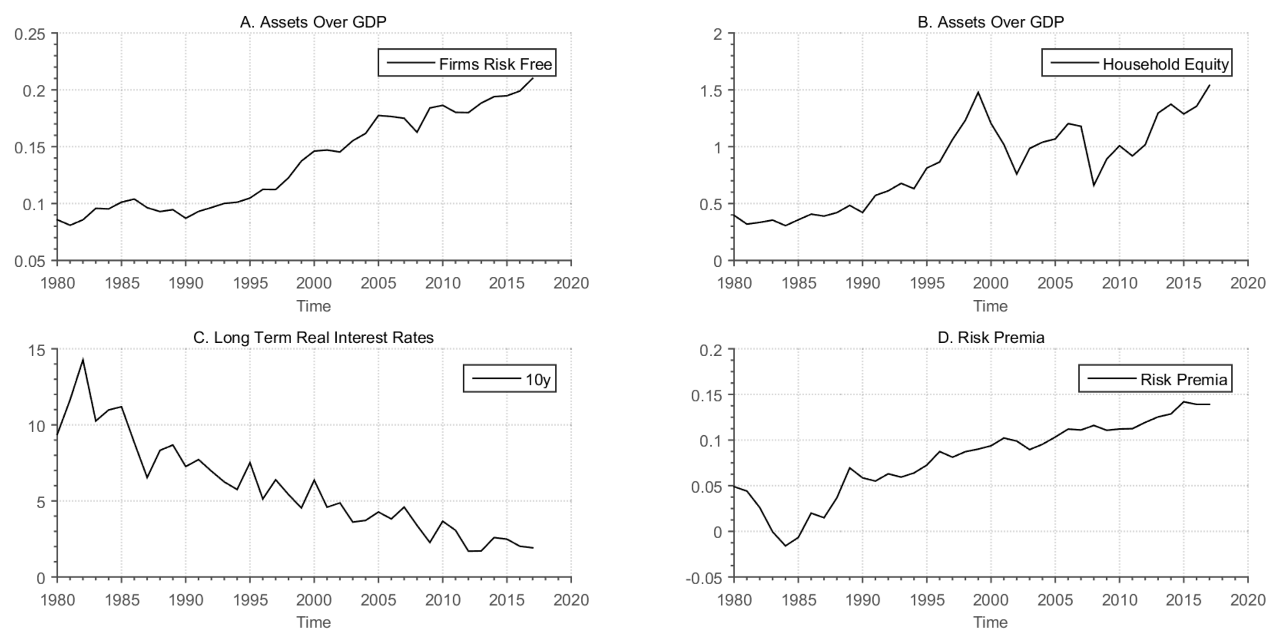
[25] This effect only shows up when one solves the model economy globally. If we had approximated the solution log linearizing around the steady state, the risk-free rate would always remain around the observed values between $0 - 10\%$. Indeed, during our stochastic simulations the risk-free rate is never above 5%. See figures 3 and 5.

Figure 3: Model generated fall in the labor share



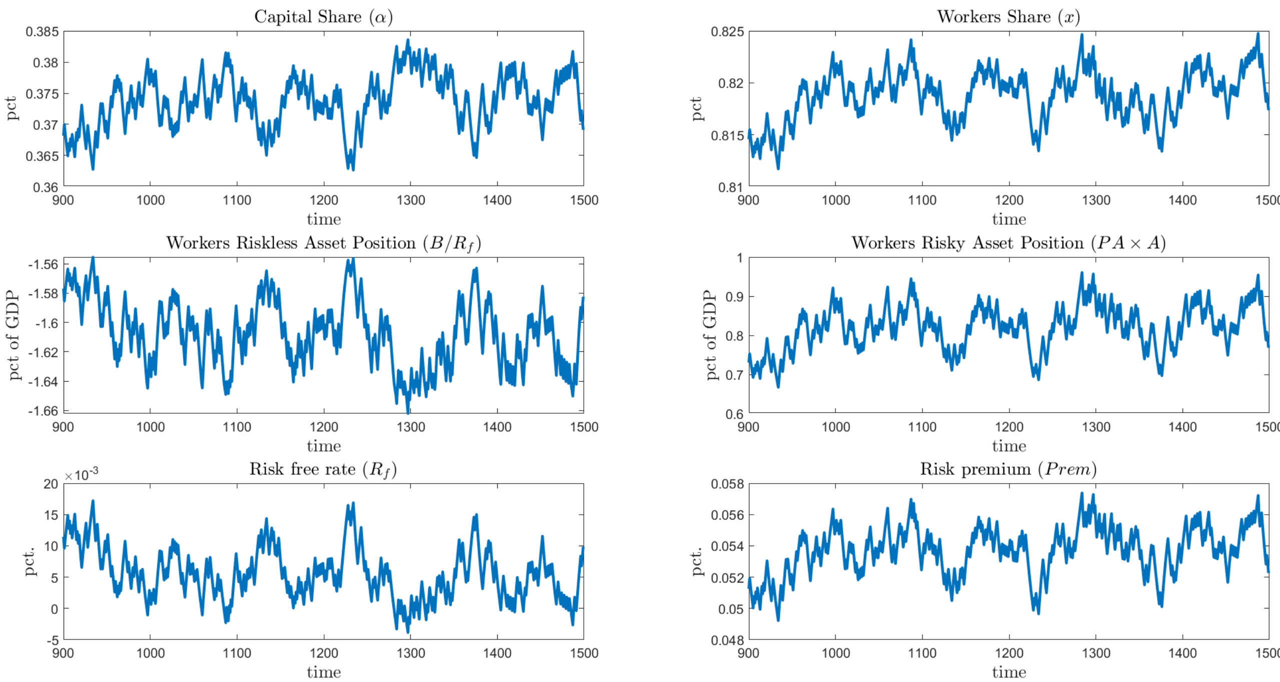
Note: This figure displays key quantities for our calibrated model generating an increasing path for α by feeding the model economy with a path of productivity shocks with all values equal to s_H.

Figure 4: Corporate savings and household equity gluts



Note: U.S. data from 1980 onwards, see Appendix B.1 for additional details.

Figure 5: Time path



Note: This figure displays key quantities for the calibrated model after a sequence of computer generated random shocks to the productivity of capital s.

to sustain the capital stock. Since positive aggregate shocks decrease x, this second effect could generate a negative correlation between the risk premium and GDP. As a result, the opposite forces between the direct and indirect effects could generate either a positive or negative correlation between the risk premium and output, depending on the strength of the wealth effects.

Are these patterns consistent with the data? In Appendix B.1 we describe in detail how, using widely available data, we construct model-consistent measures for the U.S. economy. Figure 4 shows the results for the period starting in 1980. We choose this starting point in this section for ease of comparison with other literatures that focus on a similar time frame. Nevertheless, Figure 6 of Appendix B.1 shows the equivalent calculations starting in 1946. Most of the findings are standard, except, perhaps, the increasing risk premium. However, this is not especial to this paper, since it has also been documented for the U.S. economy by Farhi and Gourio (2018), and globally by Chirinko and Mallick (2022). Indeed, our measure of risk premia uses Chirinko and Mallick (2022) methodology, but focusing in the U.S., proxying it with the difference between the marginal productivity of private capital and a risk-free rate, which maps exactly to our model's computation. These observations can be compared with Figure 3, where we show a path with a continuously increasing capital share (decreasing labor share). In short, Figure 4 shows a striking similarity in terms of patterns to those predicted by our theory. The corporate savings glut is accompanied by a households' equity glut, falling interest rates and an increase in the level of the risk premium. Thus, although we do not claim that our mechanism is THE driver of such patterns, it is certainly consistent and it maybe playing an important role.

It is also evident from Figures 3 and 4 that the levels of the calibrated quantities are relatively distant from the actual observations. We have too many risk-free assets and too few risky assets. As we mentioned before, the purpose of the exercise is not to exactly match all the moments but to assess the quantitative relevance of this mechanism. To properly match all the moments, we would need to add additional motives to hold and trade financial assets, which is beyond the scope of this paper. Nevertheless, we show that *seemingly small variations in the income shares have a large impact on the financial markets*.

The path chosen for α in Figure 3, although observed in data, may appear arbitrary within the structure of the model. Could random paths with *i.i.d* aggregate shocks generate a similar pattern? As mentioned in Section 3.3, the g_s shock is such that the *growth rates are i.i.d.*, so that the *levels* are close to random walks. Figure 5 depicts a typical path from a simulation of the model. Although the shocks are *i.i.d.*, the model generates long time spans of an increasing capital share (top left panel) and a decreasing consumers' share of wealth (top right panel). In addition, the workers' risky and risk-free asset positions (middle panels) are almost perfectly negatively correlated. This can be seen in detail in Table 3, which displays the model-implied correlations.[26] Most predictions derived analytically in Section 2, reaffirmed in Section 3.3 and clearly depicted in Figure 3 are part of the typical random paths of this economy.

Finally, it is important to note that our model economy cannot generate a negative correlation between the risk premium and GDP, which is the focus of a large fraction of the quantitative studies in macro finance (e.g. Brunnermeier and Sannikov (2014), Di Tella (2017), Gârleanu and Panageas (2015)). In our setting, the direct effect due to the increase in α cannot be overcome by opposing forces generated by changes in relative wealth.[27] Given that the focus of our paper is the medium-long term, which we match

[26] Additional correlations for other variables can be found in Table 4 in Appendix A.

[27] In online Appendix G we show how to significantly reduce the short term correlation by maximizing the impact of the wealth effects. What we do is to simulate the economy for "only" 1500 periods and start the simulation with a significantly low initial $x_0 = 0.37$. In this case we are able to replicate similar averages to those in Table 2, but by including the transition paths in the calculations (with x_t always increasing), we reduce the correlation between $\alpha(s)$ (and GDP) and the risk premium to zero. This exercise stresses the role of changes in relative wealth (not powerful enough around the steady state), and the significantly low convergence speed of x.

Table 3: Implied correlations

Capital Share	Risk Free Position	Risky Position	Risk Free Rate	Risk Premium
1.00	-1.00	0.90	-0.99	0.92
-1.00	1.00	-0.91	0.99	-0.93
0.90	-0.91	1.00	-0.95	1.00
-0.99	0.99	-0.95	1.00	-0.97
0.92	-0.93	1.00	-0.97	1.00

as shown in Figure 4, and we are abstracting from many elements affecting short term fluctuations, we believe that the correlation between output and the risk premium does not invalidate our analysis.[28] In addition, there are variations of our model that would yield a negative correlation. For instance, one could think that the increase in the risk premium during recessions could be due to the temporary increase in uncertainty, as in Bloom (2009) and Di Tella (2017). Alternatively, even though we are focusing on a positive correlation between the capital share and GDP, there is substantial evidence that in the short term this correlation is negative, see Ríos-Rull and Santaeulàlia-Llopis (2010) and León-Ledesma and Satchi (2018). Incorporating these elements into our environment would allow us to improve the model fit to financial variables. However, we believe that including these variations is beyond the scope of this paper, so we leave them for future research.

5 Conclusions

The Kaldor facts led to the prevailing belief that the capital and labor income shares were, aside from some small short-run variations, roughly constant. An important implication of this belief is the impossibility for workers and capitalists to insure each other. With constant income shares, aggregate fluctuations affect both sectors equally and only common uninsurable shocks are left. Recent studies, however, have shown that the income shares move in both the short and medium-long run.

In this paper we argue that variations in the income shares creates an important motive to share risk between capitalists and workers. Since both are differentially affected by aggregate shocks, they have incentives to trade in the financial markets to insure changes

[28] We thank an anonymous referee for suggesting this point.

in their relative income. When the labor share is counter-cyclical, the optimal insurance contract can be implemented by workers borrowing in risk-free assets and buying equity to participate in capitalists' gains. The presence of capitalists uninsured idiosyncratic risk decreases their willingness to trade, hampering the implementation of the optimal contract. Nevertheless, we show in a calibrated model that this channel is quantitatively large, to the extent that it can by itself account for some observed long-term patterns in the financial markets: the corporate savings glut, the households' equity glut, the falling interest rates and the increased risk premium. Thus, although there are certainly other factors shaping these patterns, the mechanism proposed here cannot be ignored.

The focus of this paper is on the medium-long run. However, our model would lend itself naturally to the study of how income shares exacerbate or mitigate fluctuations. The setup is also suitable for analyzing questions linking inequality and asset pricing. In particular, ours is a two-factor asset pricing model in which the capital share and the relative wealth of financial intermediaries are factors pricing the "cross-section" of assets. Both business cycles and asset prices are topics for further research.

Finally, we have abstracted from many frictions that can either directly affect financial markets, such as inflation, liquidity concerns, default risk, etc., or that can indirectly affect the financial sector through links to the real economy, such as wage and price rigidities. The interactions of these frictions with our mechanism are interesting paths for future research.

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A Additional tables

Table 4: Correlation of simulated variables - Table for the appendix of the paper

Capital Share	Risk Free Position	Risky Position	Risk Free Rate	Workers Share	Investment	Risk Premium
1.00	-1.00	0.90	-0.99	0.66	-0.38	0.92
-1.00	1.00	-0.91	0.99	-0.69	0.39	-0.93
0.90	-0.91	1.00	-0.95	0.93	-0.48	1.00
-0.99	0.99	-0.95	1.00	-0.77	0.42	-0.97
0.66	-0.69	0.93	-0.77	1.00	-0.48	0.90
-0.38	0.39	-0.48	0.42	-0.48	1.00	-0.47
0.92	-0.93	1.00	-0.97	0.90	-0.47	1.00

B Stylized facts

B.1 Post 1980's

Figure 4 plots four statistics which we consider important for empirically validating our work. Panel A shows that the share of risk-free assets in U.S. firms' portfolios has been increasing in the last 30 years. We define risk-free assets as the sum of private foreign deposits, checkable deposit and currency, total time and savings deposits, money market fund shares, security repurchase agreements, commercial paper, treasury securities, agency and GSE backed securities, municipal securities and mutual fund shares. The amount of risk-free assets is then normalized by nominal GDP. Our definition of risk-free assets is similar to the one of broad liquid assets for nonfinancial corporations given by the Board of Governors of the Federal Reserve; ours differs as we consider both financial and nonfinancial corporations and exclude corporate equity, as we will treat equity differently from other liquid assets. The indicator on corporate risk-free debt has received a lot of attention in the recent years because of what is now known as the "Corporate savings glut." This pattern is not peculiar to the U.S. but it is evident in many countries. [29]

Panel B shows an increase in the share of risky assets in households' portfolios as a proportion of GDP. Indeed, households' equity holdings have almost tripled from 1980

[29] This type of measure is used for example in the empirical literature on the determinants of corporate cash holdings (see Opler et al. (1999)) and in the literature on intangible capital. See Chen et al. (2017) for a characterization of patterns of sectoral saving and investment for a large set of countries over the past three decades. Among other possibilities, increased net lending can be associated with accumulation of cash, repayment of debt, or increasing equity buybacks net of issuance, as highlighted by Bates et al. (2009) and Foley et al. (2007).

to 2014, with a clear upward trend. To compute the series, we normalize the amount of households' directly and indirectly held corporate equities over nominal GDP. This kind of indicator is generally considered for cross sectional analyses in the household finance literature.[30]

Panel C depicts the 10-year U.S. real interest rate, which has been falling continuously since the '80s. This feature has been widely documented; explanations for this falling rate range from demographics, to passing from a secular stagnation, to a sudden increase in uncertainty.[31] Long term nominal interest rates and the consumer price index are taken from the FRED database.

Panel D plots the gap between the marginal productivity of private capital and the risk free rate. To this end we use the methodology by Chirinko and Mallick (2022) using data from the Penn World tables 9.0, IMF investment and capital stock database and the World Bank data for the natural capital share. As the risk free rate we use the U.S. Treasury Securities at 10-Year Constant Maturity from FRED. We use long rate because the maturity is consistent with that of most productive investments. In addition, the fact that we use Penn World Tables is makes the methodology extendable to other countries. Our results are in line with the findings of Farhi and Gourio (2018), which uses an alternative methodology arrive to the same pattern for the U.S. economy.

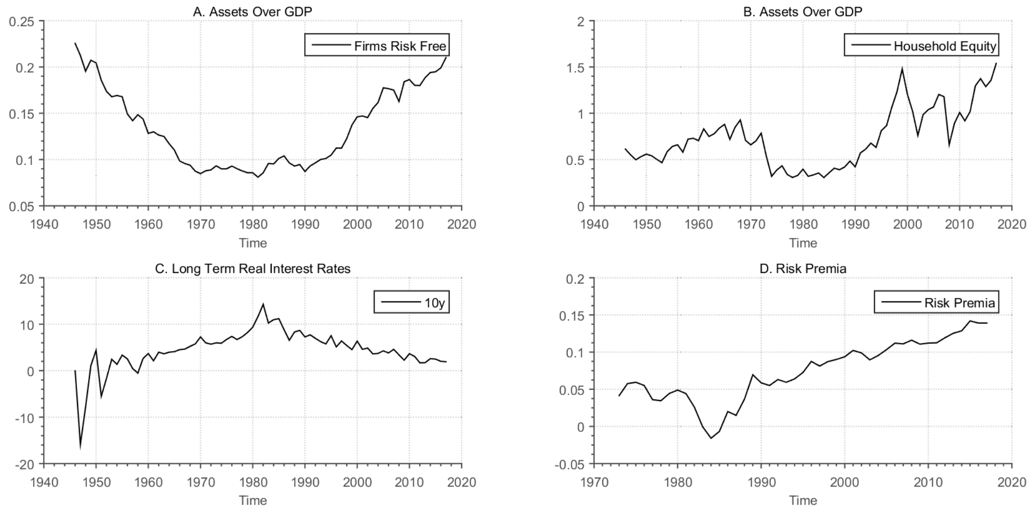
Post WWII and Data Sources

Long-term interest rates. Long-term interest rates are taken from two sources. First, we use FRED data from 1962 to 2017, series DGS10, source Board of Governors of the Federal Reserve System (US), release: H.15 Selected Interest Rates. Units are in percent, not seasonally adjusted, at a monthly frequency. The rates are averages of business days. Inflation is computed from the Consumer Price Index for all urban consumers series CPIAUCSL also taken from FRED. The index series is sampled at monthly frequency, with base year 1982-1984, seasonally adjusted. Data before 1962 are taken from Robert Shiller's update of data shown in Chapter 26 of Shiller (1992), and Shiller (2015). For the long term interest rate, the author uses the 10-year Treasury after 1953; before 1953, it is government bond yields from Homer and Sylla (1996). Shiller uses the CPI (Consumer Price Index-All Urban Consumers) published by the U.S. Bureau of Labor Statistics. We compute the monthly inflation as the percentage change of the consumer price index and obtain the

[30]See Campbell (2006). Most papers use SCF data to look at these statistics (Bergstresser and Poterba (2004), Bertaut and Starr (2000), Heaton and Lucas (2000), Poterba and Samwick (2001), while we use the Flow of Funds data to construct a longer series that we show Appendix B.1.

[31]See for example Karabarbounis and Neiman (2014), Caballero et al. (2017), Carvalho et al. (2016) and Summers (2014).

Figure 6: Saving and equity gluts



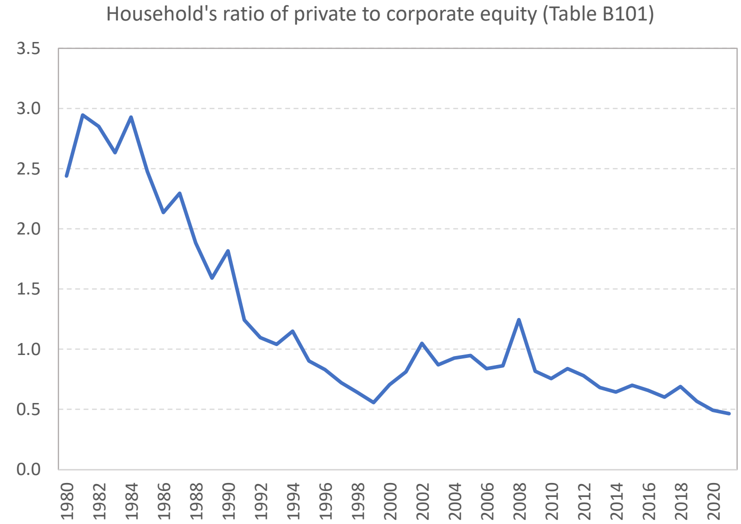
Note: U.S. Data. Data are from 1946 onwards, except for the risk premium that starts on 1970.

real 10-year rate by subtracting the inflation, times 100, from the nominal interest rate.

Assets. Data are from the Board of Governors of the Federal Reserve system. Firms' risk-free debt is defined as the sum of private foreign deposits, checkable deposit and currency, total time and savings deposits, money market fund shares, security repurchase agreements, commercial paper, treasury securities, agency and gse backed securities, municipal securities and mutual fund shares. All the series are in table L.102 Nonfinancial Business, Billions of dollars; amounts outstanding end of period, not seasonally adjusted. To compute household equity we use the B.101.e Balance Sheet of Households and Nonprofit Organizations with Equity Detail, Billions of dollars; amounts outstanding end of period, not seasonally adjusted, precisely the entry Directly and indirectly held corporate equities (serie LM153064475). Both quantities are normalized over nominal GDP.

Private and public equity

Figure 7: Fall in the private equity ratio



Note: U.S. Data. Data are from 1980 onwards. The figure shows the ratio of private equity holdings by households (Line 28, Table B.101, Flow of Funds tables), over corporate equity (Line 28, Table B.101, Flow of Funds tables).

C Proofs

C.1 Profits are linear in the effective capital stock.

This result is valid for both the two period model and the infinite horizon economy. Recall that the technology is:

$$y(s,i,k,l) = F(g_i g_s k, l) + (1-\delta) g_i g_s k_i$$

where F is homogeneous of degree one. The profits are given by:

$$\pi(s,i,k,l) = \max_l \{F(g_i g_s k, l) + (1-\delta) g_i g_s k_i - \omega(s) l\}$$

Because the technology is CRS, we can write it as:

$$\begin{aligned}
\pi(s,i,k,l) &= \left[\max_{\frac{l}{g_i g_s k_i}} \left\{ F(1, \frac{l}{g_i g_s k_i}) + (1-\delta) - \frac{\omega(s) l}{g_i g_s k_i} \right\} \right] g_i g_s k \\
&= \left[\max_{\tilde{l}} \{F(1,\tilde{l}) + (1-\delta) - \omega(s)\tilde{l}\} \right] g_i g_s k \\
&= r(\omega(s)) g_s g_i k \\
&= r(s) g_i k_i
\end{aligned}$$

where we have defined:

$$r(\omega(s)) \equiv \left[\max_{\tilde{l}} \left\{F(1,\tilde{l}) + (1-\delta) - \omega(s)\tilde{l}\right\}\right]$$

Thus, the shock to capital, including the effect in depreciation, renders the problem linear in individual capital and the idiosyncratic shock. As mentioned before the gross return $r(s)$ includes the depreciation rate. The net return on capital is $r^n(s) \equiv r(\omega(s)) - (1-\delta)$.

C.2 Proof of Proposition 1

For part (a), we first characterize the equilibrium of the two-period model as the functions $\{\phi(s), \hat{p}(s)\}$ where $\hat{p}(s) = p(s)g_s$ is defined as growth-adjusted Arrow-Debreu prices. We then characterize the welfare shares in the equilibrium. Finally, we solve for asset prices and quantities:

Part (a): Asset Prices and Quantities. *Step 1: Characterization of the Equilibrium.* Recall equation (9):

$$\frac{u'(e_1)}{u'(c_1)} = \frac{\mathbb{E}_i[(-\phi(s)Y_2(s) + \alpha(s)Y_2(s)g_i)^{-\sigma}]}{(\phi(s)Y_2(s) + (1-\alpha(s))Y_2(s))^{-\sigma}}.$$

Replacing the budget constraint in the period 1 by the individuals consumptions, dividing all the components in period 1 by Y_1, cancelling $Y_2(s)$ in the right hand side of (9), and using the definition of $\hat{p}(s)$ the last equation can be written as:

$$\left(\frac{y_1 + \sum_s \hat{p}(s)\phi(s)}{1 - y_1 - \sum_s \hat{p}(s)\phi(s)}\right)^{-\sigma} = \frac{\mathbb{E}_i[(-\phi(s) + \alpha(s)g_i)^{-\sigma}]}{(\phi(s) + (1-\alpha(s)))^{-\sigma}}$$

where $y_1 = \alpha_1 + E_1/Y_1$ is the share of resources in hands of entrepreneurs in period 1. Using the definition of consumption equivalent in (11), and taking into account that the Arrow-Debreu prices satisfy $p(s) = \Pi(s)\frac{\mathbb{E}_i u'(e_2(s,i))}{u'(e_1)}$, we can characterize the equilibrium as the functions $\{\phi(s), \hat{p}(s)\}$ satisfying:

$$\frac{y_1 + \sum_s \hat{p}(s)\phi(s)}{1 - y_1 - \sum_s \hat{p}(s)\phi(s)} = \frac{-\phi(s) + \alpha(s)g^{ce}(s)}{\phi(s) + (1-\alpha(s))}. \tag{42}$$

$$\hat{p}(s) = \Pi(s)\left(\frac{y_1 + \sum_s \hat{p}(s)\phi(s)}{-\phi(s) + \alpha(s)g^{ce}(s)}\right)^{\sigma} g_s^{1-\sigma}. \tag{43}$$

Step 2: Welfare Shares. Notice that $\sum_s \hat{p}(s)\phi(s)$ are the total entrepreneur's savings in units of time 1 output. Thus $y_1 + \sum_s \hat{p}(s)\phi(s)$ is the normalized consumption of the capitalist. With these definitions the total economy's wealth in units of period 1 output is $1 + \sum_s \hat{p}(s)$. Similarly, the worker's present value of resources, normalized by Y_1, is: $1 - y_1 + \sum_s \hat{p}(s)(1 - \alpha(s))$. Therefore, the equilibrium initial wealth ratio is:

$$x_1 = \frac{1 - y_1 + \sum_s \hat{p}(s)(1 - \alpha(s))}{1 + \sum_s \hat{p}(s)}. \tag{44}$$

Step 3: A System for $p(s)$ and $\phi(s)$: Firstly we derive the equation for $p(s)$. To simplify notation define:

$$P = \sum_s \hat{p}(s) \in \mathbb{R}_+; \quad \text{and} \quad P_\phi = \sum_s \hat{p}(s)\phi(s) \in \mathbb{R}.$$

Operating with equation (42) we can write:

$$\phi(s) = \alpha(s) g^{ce}(s)[1 - y_1 - P_\phi] - (1 - \alpha(s))[y_1 + P_\phi]; \;\; \forall s. \tag{45}$$

Multiplying the last by $\hat{p}(s)$ and adding up we obtain:

$$P_\phi = \sum_s \hat{p}(s)\alpha(s) - P[y_1 + P_\phi] + \sum_s \hat{p}(s)\alpha(s)[g^{ce}(s) - 1][1 - y_1 - P_\phi].$$

Adding y_1 in both sides of the last equation and reorganizing generates:

$$y_1 + P_\phi = \frac{y_1 + \sum_s \hat{p}(s)\alpha(s)}{1 + P} + \frac{\sum_s \hat{p}(s)\alpha(s)[g^{ce}(s) - 1][1 - y_1 - P_\phi]}{1 + P}. \tag{46}$$

$$= 1 - x_1 + \hat{c} \sum_s \hat{p}(s)\alpha(s)[g^{ce}(s) - 1]. \tag{47}$$

where $\hat{c} = \frac{1 - y_1 - P_\phi}{1 + P} \geq 0$. Hence, equations (45) and (47) generate the solution for the quantities of transacted assets.

Step 4: Asset Prices: To solve for the prices note that equation (45) can be rewritten as:

$$[y_1 + P_\phi][(1 - \alpha(s)) + \alpha(s) g^{ce}(s)] = -\phi(s) + \alpha(s) g^{ce}(s).$$

Using the last relationship in the price equation (43) and recalling that $\hat{p}(s) = p(s) g_s$ we obtain:

$$p(s) = \Pi(s) \left(1 + \alpha(s)[g^{ce}(s) - 1]\right)^{-\sigma} g_s^{-\sigma}. \tag{48}$$

Step 5: Asset Quantities: For $\phi(s)$, reorganize (45):

$$\phi(s) = \alpha(s)g^{ce}(s) - (y_1 + P_\phi) + \alpha(s)[1 - g^{ce}(s)][y_1 + P_\phi].$$

Plugging (47) in the last:

$$\phi(s) = \alpha(s)g^{ce}(s) + [\alpha(s)(1 - g^{ce}(s)) - 1][1 - x_1 + \hat{c}\sum_s \hat{p}(s)\alpha(s)[g^{ce}(s) - 1]].$$

Define:

$$\Gamma(g^{ce}) := -\hat{c}\sum_s \hat{p}(s)\alpha(s)[g^{ce}(s) - 1]] > 0.$$

This holds because $\mathbb{V}ar(g_i) > 0$ implies that $g^{ce}(s) < 1$ for all s, then:

$$\phi(s) = \alpha(s)g^{ce}(s) + [\alpha(s)(1 - g^{ce}(s)) - 1][1 - x_1 - \Gamma(g^{ce})],$$

$$\phi(s) = x_1 - (1 - \alpha(s)) + \alpha(s)(g^{ce}(s) - 1)[x_1 + \Gamma(g^{ce})] + \Gamma(g^{ce}). \tag{49}$$

which is (13) in Proposition 1.

Part (b): Evolution of the wealth shares. Since the second period is the last one, wealth ratio in that period equals to the income ratio which is $x_2(s) = \phi(s) + 1 - \alpha(s)$. Plugging (49) to this expression and substracting x_1 gives:

$$x_2(s) - x_1 = \alpha(s)(g^{ce}(s) - 1)[x_1 + \Gamma(g^{ce})] + \Gamma(g^{ce}).$$

which is (14) that describes the evolution of wealth shares.

Part (c): Precautionary Savings. Recall that $y_1 + P_\phi$ is the capitalist's normalized consumption and also its consumption share out of output. When $\mathbb{V}ar(g_i) = 0$ i.e. $g^{ce} = 1; \forall s$, from (47), we have $y_1 + P_\phi = 1 - x_1$ so that the consumption share is equal to the wealth ratio. When $\mathbb{V}ar(g_i) > 0$ we have $g^{ce} < 1; \forall s$. Therefore, $y_1 + P_\phi < 1 - x_1$. Due to the presence of uninsured idiosyncratic risk, capitalists consume a smaller proportion of their wealth hence save more. Using (47) we can state this formally. Let $\sum_s p(s)\phi(s)$ and $\sum_s p(s)^{CM}\phi(s)^{CM}$ be P_ϕ in $\mathbb{V}ar(g_i) > 0$ and $\mathbb{V}ar(g_i) = 0$ cases respectively. Then:

$$\sum_s p(s)\phi(s) - \sum_s p(s)^{CM}\phi(s)^{CM} = \hat{c}\sum_s \hat{p}(s)\alpha(s)[g^{ce}(s) - 1] < 0$$

which proves part (c) of the proposition.

C.3 Proof of Proposition 2

Recall that the equation for asset quantities from Proposition 1:

$$\phi(s) = x_1 - (1-\alpha(s)) + \alpha(s)(g^{ce}(s)-1)[x_1 + \Gamma(g^{ce})] + \Gamma(g^{ce}). \tag{50}$$

In equilibrium $A^w + A^e = 0$ and $B^w + B^e = 0$. Note that there is a one-to-one mapping between the second period's payoffs of the AD securities $\phi(s)$ and the payoffs of a portfolio with the assets:

$$\phi(L)Y_2(L) = R_L B^w + A^w \alpha(L) Y_2(L)$$
$$\phi(H)Y_2(H) = R_L B^w + A^w \alpha(H) Y_2(H)$$

The latter implies positions and prices given by

$$A^w = \frac{\phi(H)Y_2(H) - \phi(L)Y_2(L)}{\alpha(H)Y_2(H) - \alpha(L)Y_2(L)} \tag{51}$$
$$R_L B^w = \frac{Y_2(L)Y_2(H)(\phi(L)\alpha(H) - \alpha(L)\phi(H))}{\alpha(H)Y_2(H) - \alpha(L)Y_2(L)} \tag{52}$$
$$R_L = \frac{1}{\sum_s p(s)} \tag{53}$$
$$P_A = \sum_s p(s)\alpha(s)Y_2(s). \tag{54}$$

Where $p(s)$ is the price of the AD securities, then both the consumer and the entrepreneur are optimizing. Evaluating (50) in high and low states and plugging the respective values to respective places in (51) and (52) we have:

$$A^w = 1 - \left(\frac{Y_2(H) - Y_2(L)}{\pi_2(H) - \pi_2(L)}\right)(1-x_1) + x_1\left[\frac{\alpha(H)Y_2(H)g^{CE}(H) - \alpha(L)Y_2(L)g^{CE}(L)}{\pi_2(H) - \pi_2(L)} - 1\right] + \mathbf{\Xi} \tag{55}$$

where $\mathbf{\Xi} = \Gamma \times \frac{Y_2(H)\left(\alpha(H)\left(g^{CE}(H)-1\right)+1\right) - Y_2(L)(\alpha(L)\left(g^{CE}(L)-1\right)+1)}{\pi_2(H)-\pi_2(L)}$.

$$R_L B^w = -\left(\frac{\alpha(H) - \alpha(L)}{\pi_2(H) - \pi_2(L)}\right)Y_2(L)Y_2(H)(1-x_1) - \frac{Y_2(L)Y_2(H)\alpha(H)\alpha(L)}{\pi_2(H) - \pi_2(L)}(g^{CE}(H) - g^{CE}(L))x_1 + \mathbf{\Psi} \tag{56}$$

where $\mathbf{\Psi} = \Gamma \times \frac{\alpha(H)\alpha(L)\left(g^{CE}(L)-g^{CE}(H)\right)+\alpha(H)-\alpha(L)}{(Y_2(L)Y_2(H))^{-1}(\pi_2(H)-\pi_2(L))}$.

C.4 Lemma: If $A_1 \geq 0$ then $x_1^{IM} \leq x_1^{CM}$

Suppose $A_1 = 0$, $\mathbb{E}[\alpha(s)] \geq \alpha_1$ and $Y(s) = Y_1$ for all s. **Step 1**. Under these assumptions we can write equation (10) as:

$$x_1 = \frac{(1-\alpha_1)Y_1 + \sum_s p(s)(1-\alpha(s))Y(s)}{Y_1 + \sum_s p(s)Y(s)} = 1 - \frac{\sum_s p(s)\alpha(s)g_s + \alpha_1}{1+\sum_s p(s)g_s}. \tag{57}$$

Note that from Proposition 1 $p(s)$ can be written as $p(s) = \Pi(s)D(s)g_s^{-\sigma}$, where $D(s) := [1+\alpha(s)\left(g^{ce}(s)-1\right)]^{-\sigma}$, so the last equation becomes:

$$x_1 = 1 - \frac{\mathbb{E}[D(s)g_s^{-\sigma}]\mathbb{E}[\alpha(s)] + Cov(\alpha(s), D(s)g_s^{-\sigma}) + \alpha_1}{1+\mathbb{E}[D(s)g_s^{-\sigma}]}.$$

From equation (12), $D(s) = [1+\alpha(s)(g^{ce}(s)-1)]^{-\sigma} \geq 1$ for all s. We can show that $Cov(\alpha(s), D(s)) > 0$. **Step 2**. Note that under complete markets $\mathbb{E}[D(s)g_s^{-\sigma}] = \mathbb{E}[g_s^{-\sigma}]$ and $Cov(\alpha(s), D(s)g_s^{-\sigma}) = Cov(\alpha(s), g_s^{-\sigma})$.

$$x_1^{CM} = 1 - \frac{\mathbb{E}[g_s^{-\sigma}]\mathbb{E}[\alpha(s)] + Cov(\alpha(s), g_s^{-\sigma}) + \alpha_1}{1+\mathbb{E}[g_s^{-\sigma}]}$$

Step 3. As a result $x_1^{IM} \leq x_1^{CM}$ iff:

$$\frac{\mathbb{E}[D(s)g_s^{-\sigma}]\mathbb{E}[\alpha(s)] + Cov(\alpha(s), D(s)g_s^{-\sigma}) + \alpha_1}{1+\mathbb{E}[D(s)g_s^{-\sigma}]} \geq \frac{\mathbb{E}[g_s^{-\sigma}]\mathbb{E}[\alpha(s)] + Cov(\alpha(s), g_s^{-\sigma}) + \alpha_1}{1+\mathbb{E}[g_s^{-\sigma}]}$$

Using the definition of covariance, it holds that:[32]

$$\mathbb{E}[D(s)g_s^{-\sigma}] = \mathbb{E}[D(s)]\mathbb{E}[g_s^{-\sigma}] + Cov[D(s), g_s^{-\sigma}].$$

Our main assumption is that capital shares are pro-cyclical, i.e., $Cov(\alpha(s), g_s) \geq 0$, which implies that $Cov[D(s), g_s^{-\sigma}] \leq 0$ and that $Cov(\alpha(s), g_s^{-\sigma}) \leq 0$. Then $x_1^{IM} \leq x_1^{CM}$ iff:

$$\mathbb{E}[g_s^{-\sigma}]\left(\mathbb{E}[\alpha(s)] - \alpha_1\right)\left(\mathbb{E}[D(s)] + \frac{Cov[D(s), g_s^{-\sigma}]}{\mathbb{E}[g_s^{-\sigma}]} - 1\right) + (1+\mathbb{E}[g_s^{-\sigma}])Cov(\alpha(s), D(s))$$

$$\geq (1+\mathbb{E}[D(s)g_s^{-\sigma}])Cov(\alpha(s), g_s^{-\sigma}) + \mathbb{E}[g_s^{-\sigma}]\mathbb{E}[\alpha(s)]Cov[D(s), g_s^{-\sigma}].$$

[32] The simplest case is when output is constant, so $g_s = 1, \forall s$, then the last is true whenever:

$$\left(\mathbb{E}[\alpha(s)] - \alpha_1\right)\left(\mathbb{E}[D(s)] - 1\right) + 2Cov(\alpha(s), D(s)) \geq 0$$

Alternatively:

$$(\mathbb{E}[\alpha(s)] - \alpha_1) \left(\mathbb{E}[D(s)g_s^{-\sigma}] - \mathbb{E}[g_s^{-\sigma}]\right) + (1 + \mathbb{E}[g_s^{-\sigma}])Cov(\alpha(s), D(s))$$

$$\geq (1 + \mathbb{E}[D(s)g_s^{-\sigma}])Cov(\alpha(s), g_s^{-\sigma}) + \mathbb{E}[g_s^{-\sigma}]\mathbb{E}[\alpha(s)]Cov[D(s), g_s^{-\sigma}].$$

C.5 Capital share in the CES production function

The firms maximizes $\pi(s,i) = \left[\alpha\left(g_i g_s k\right)^{\frac{\rho-1}{\rho}} + (1-\alpha)L^{\frac{\rho-1}{\rho}}\right]^{\frac{\rho}{\rho-1}} - \omega L$, which implies $L^d(s,i) = \alpha^{\frac{\rho}{\rho-1}}\left[\left(\frac{\omega}{1-\alpha}\right)^{\rho-1} - (1-\alpha)\right]^{\frac{\rho}{1-\rho}} g_i g_s k$. From the labor market clearing condition $1 = L^s = L^d(s) = \mathbb{E}(L^d(s,i))$ we obtain the wage:

$$\omega(s) = (1-\alpha)\left[\alpha\left(g_s k\right)^{\frac{\rho-1}{\rho}} + (1-\alpha)\right]^{\frac{1}{\rho-1}}$$

Moreover recall that:

$$\alpha(s,i) = \frac{\partial y(s,i)}{\partial k}\frac{k}{y(s,i)} = \frac{\alpha(g_i g_s k)^{\frac{\rho-1}{\rho}}}{\alpha(g_i g_s k)^{\frac{\rho-1}{\rho}} + (1-\alpha)(L)^{\frac{\rho-1}{\rho}}}$$

so $\alpha(s) = \mathbb{E}_i(\alpha(s,i))$ is given by:

$$\alpha(s) = \frac{\alpha(g_s k)^{\frac{\rho-1}{\rho}}}{\alpha(g_s k)^{\frac{\rho-1}{\rho}} + (1-\alpha)}$$

and, given that $Y(s) = \mathbb{E}(y(s,i))$, in the same way, the labor share is:

$$(1-\alpha(s)) = \frac{\partial Y}{\partial L}\frac{L}{Y} = \frac{(1-\alpha)}{\alpha(g_s k)^{\frac{\rho-1}{\rho}} + (1-\alpha)}$$

Then $\omega(s) = (1-\alpha(s))Y(s)$. Given the wage, $L^d(s,i) = g_i$ and therefore $\pi(s,i) = \alpha(s)Y(s)g_i$.

D Numerical Appendix

a. Using $g_s k$, compute once and for all $Y(s)$, $r(s)$ and $w(s)$.

b. Step A: Guess and Solve. Guess the functions $\left\{\ p(s'|s), \nu(s), \frac{R^{\frac{1}{\gamma}}}{1+Prem(s)}, \Pi(s'|s)\ \right\}$.

We start assuming a constant investment, a uniform distribution for Π and we assume that $p(s'|s) = \beta\Pi(s'|s)\left[\frac{Y(s')}{Y(s)}\right]^{-\gamma}$.

Using equations (25), (32) and

$$\tilde{\beta}\left(s', s\right) = \frac{W^T(s')/W^T(s)}{\left(\frac{\zeta(s)(1-\zeta(s'))}{(1-\zeta(s))}\right)^{\frac{1-\gamma}{\gamma(1-1/\sigma)}}\zeta(s)x + \left(\frac{\beta^e}{\beta}\right)^{\frac{(1-\gamma)}{\gamma(1-\sigma)}}\left(\frac{\vartheta(s)(1-\vartheta(s'))}{(1-\vartheta(s))}\right)^{\frac{1-\gamma}{\gamma(1-1/\sigma)}}\mathbb{R}(s'|s)^{\frac{1}{\gamma}}}$$

we obtain the savings rates for the consumer, the entrepreneur, and $\tilde{\beta}\left(s', s\right)$. To this end, note that from equation (82) in the Online Appendix F, the level of wealth consistent with the equilibrium savings ratios and investment choice can be written as:

$$W^T(s) = \frac{Y(s)}{(1-\zeta(s))x + [1-\vartheta(s)(1-\nu(s))](1-x)}$$

Thus, using the last two equations we simultaneously solve for ϑ and ζ. We do so by appealing to its recursivity. We guess initial function $\vartheta(s) = \zeta(s) = \beta$ and iterate until convergency. We obtain ϕ^w and ϕ^e from (24) and (31).

c. Step B: Update $\left\{\ p(s'|s), \nu(s), \frac{R(s'|s)^{\frac{1}{\gamma}}}{1+Prem(s)}, \Pi(s'|s)\ \right\}$. To update $\nu(s)$, we obtain investment k':

$$k'(s) = \sum_{s'|s} \frac{p\left(s' \mid s\right)\left[\alpha\left(s'\right) + (1-\delta)g_{s'}\right]y\left(s'\right)}{(1+\mathrm{Prem}(s))}$$

and the implied $\nu(s)$ from (30). To obtain $Prem(s)$ we use (41) and to update $\mathbb{R}\left(s'|s\right)$ we use:

$$\mathbb{R}\left(s'|s\right) = 1 + \left[\frac{\gamma(1+\gamma)\left(v(s)r\left(s'\right)\right)^2 \mathrm{Var}\left(g_i\right)}{2o\left(s', 1; \phi', v\right)^2}\right].$$

The transition matrix $\Pi(s'|s)$ is then given by (39) and (40). We do so by computing the linear interpolation basis of k' in the k grid, which togethers with the transition probabilities of the exogenous shock generate a stochastic matrix.

Finally, note that we have $\tilde{\beta}\left(s', s\right)$, and by its definition:

$$p(s'|s) = \beta^{\frac{(1-\gamma)}{(1-\sigma)}}\Pi(s'|s)\tilde{\beta}(s', s)^{-\gamma}$$

We use this equation to update the price function.

d. We iterate steps A and B until convergence.

Online Appendix to "The Macroeconomics of Hedging Income Shares"

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E Financial Contract

E.1 Optimal Contract

In this section we provide the main idea for the optimal contract. Suppose that there is one period. There is a risk neutral principal which can provide insurance to the entrepreneurs. Suppose there are three possible idiosyncratic shocks $g_L < g_M < g_H$, with probability p_i.[33] At the beginning of the period, before knowing the realization of g_i, the firm can enter an insurance contract with the financial intermediary. The firm's profits are $\alpha Y g_i$. In absence of insurance the entrepreneur's utility is:

$$\mathbb{E}_i[u(e_i)] = \mathbb{E}_i[u(\alpha Y g_i)]$$

The principal (financial intermediary) can sign a contract and offer insurance to the entrepreneur.

Full Insurance. In the benchmark when g_i is observable the contract is simple: the principal "buys" all the proceeds of the production with a lump sum payment of J. Then, after the shock is realized the entrepreneur hands over the profits to the principal. Because the principal must break-even, it must be the case that $\mathbb{E}_i(\alpha Y g_i) - J = 0$. Thus, the utility of the entrepreneur in this case with full insurance is:

$$\mathbb{E}_i[u(e_i^O)] = E_i[u(J)] = u(\mathbb{E}_i[\alpha Y g_i]) > \mathbb{E}_i[u(\alpha Y g_i)] = \mathbb{E}_i[u(e_i)]$$

Moral Hazard. However, the entrepreneur is subject to moral hazard problem because g_i *is not observable*. The entrepreneur can report an alternative value of g_i, say $g_{i'}$, and keep the difference for herself. Therefore, any contract adds a constraint so that the entrepreneurs reveals the true realization of g_i (incentive compatibility). But, transforming

[33] The contract with three shocks can be generalized to any finite numbers of shocks. With only 2 shocks the results might not generalize to more states.

these "stolen" profits into consumption is not for free. Each unit of stolen profit transforms into consumption at the rate $0 \leq \psi \leq 1$. Thus, when the entrepreneur steals profits, she obtains an additional consumption of only $\psi\alpha Y(g_i - g_{i'})$. To force truthful revelation the principal must hand over additional payments d_i contingent on the realization of g_i. Since the entrepreneur will not lie in equilibrium, her consumption is $e_i^C = J + d_i$, while because the principal must break even the contract must also satisfy: $\mathbb{E}_i(\alpha Y g_i - d_i) - J = 0$, where we normalize the outside option of the principal to zero without loss of generality. As a result, the optimal contract solves:

$$\max_{\{J,d_i\}} \mathbb{E}_i u(J + d_i)$$

$$\text{st.} \quad \psi\alpha Y(g_i - g_{i'}) + d_{i'} + J \leq J + d_i; \quad \forall i, i'$$

$$\mathbb{E}_i(\alpha Y g_i - d_i) - J = 0.$$

The first set of constraints are the *incentive compatibility*, or truth telling, constraints. Notice that only the adjacent constraints matter. To see this, consider that the entrepreneur would never lie when she observes the low shock. So, only the following can be binding:

$$\begin{aligned} \psi\alpha Y(g_H - g_M) + d_M &\leq d_H \\ \psi\alpha Y(g_M - g_L) + d_L &\leq d_M \\ \psi\alpha Y(g_H - g_L) + d_L &\leq d_H \end{aligned}$$

Adding the first two inequalities:

$$\begin{aligned} \psi\alpha Y(g_H - g_M) + d_M + \psi\alpha Y(g_M - g_L) + d_L \leq &\quad d_H + d_M \\ \psi\alpha Y(g_H - g_L) + d_L \leq &\quad d_H \end{aligned}$$

Thus, the third constraint is irrelevant. In general this is a version of the single crossing property, and it can be generalized to any arbitrary number of idiosyncratic shocks. Rewriting the problem we have:

$$\max_{\{J,d_i\}} \sum_i p_i u(J + d_i)$$

$$\text{st.} \quad \psi\alpha Y(g_H - g_M) + d_M \leq d_H$$

$$\psi\alpha Y(g_M - g_L) + d_L \leq d_M$$

$$\sum_i p_i(\alpha Y g_i - d_i) - J = 0$$

Let λ be the multiplier in the break even constraint and μ_i the multiplier in each incentive compatibility. Taking first order conditions:

$$\begin{aligned}
\sum_i p_i u'(J + d_i) &= \lambda \\
\gamma_L u'(J + d_L) &= \gamma_L \lambda + \mu_M \\
\gamma_M u'(J + d_M) &= \gamma_M \lambda + \mu_H - \mu_M \\
\gamma_H u'(J + d_H) &= \gamma_H \lambda - \mu_H
\end{aligned}$$

It is clear that $\mu_L = \mu_H = 0$ cannot be a solution because it violates the IC constraints. Now, suppose $\mu_M = 0$, while $\mu_H > 0$. Then it must be that $d_L = d_M$. If $d_L > d_M$, the IC constraint implies

$$\psi \alpha Y(g_M - g_L) + d_L - d_M < 0$$

Which is a contradiction. If $d_L < d_M$ a small increase in d_L accompanied by a small reduction on d_M, keeping the break even constraint satisfied, generates a welfare change of:

$$\gamma_L d_L[u'(e_L) - u'(e_M)] > 0$$

which is true because $u''(.) < 0$ and $e_L < e_M$, thus increasing welfare. A similar argument can be used to show that $\mu_M > 0$ and $\mu_H = 0$ is not possible either. As a result, because μ_M and μ_H are both strictly positive, we must have:

$$\begin{aligned}
\psi \alpha Y(g_H - g_M) &= d_H - d_M \\
\psi \alpha Y(g_M - g_L) &= d_M - d_L
\end{aligned}$$

It is easy to see that $d_i = \psi \alpha Y g_i$, together with $J = (1 - \psi)\mathbb{E}_i(\alpha Y g_i)$, is a solution for all the equations. And since the problem has a unique solution, it must be the solution. This contract can be interpreted as an equity contract. Each entrepreneur sells a share $1 - \psi$ of her firm to the intermediary and uses the proceeds to buy an indexed stock market financial instrument. This completely smooths out a proportion $(1 - \psi)$ of the idiosyncratic risk. However, to prevent stealing not all the shares can be sold, the entrepreneur must retain a proportion ψ of her shares, which is her "*skin in the game*". This is the best insurance possible with only short term contracts. Note that here we assume that there was no aggregate risk. This result would not be affected by it, since it would affect all the IC constraints proportionally. It would only change the pricing of J.

E.2 Constrained Efficiency

In this section we show that the equilibrium in the two period model is constrained efficient. The notion of constrained efficiency follows Geanakoplos and Polemarchakis (1986) and Stiglitz (1982), and provides the planner with the same instruments as the market. In particular, the planner can intervene redistributing consumption across aggregate states with a lump sum transfer $T(s)$. Consumption for the consumer and the entrepeneur are given by:

$$\begin{aligned} c_2(s) &= T(s) + (1-\alpha(s))Y_2(s) \\ e_2(s,i) &= -T(s) + \alpha(s) g_i Y_2(s). \end{aligned}$$

Without loss of generality, and to follow the notation of the paper, we define

$$T(s) := \frac{\phi(s)}{Y_2(s)}.$$

Planning Program. The planner solves

$$\max_{\{e_1,c_1,\phi(s),c_2(s)\}_{s\in S}} \frac{e_1^{1-\gamma}}{1-\gamma} + \mathbb{E}_{i,s}\frac{e_2(s,i)^{1-\gamma}}{1-\gamma}$$

$$c_1 + e_1 = Y_1 \tag{58}$$

$$c_2(s) + e_2(s) = Y_2(s) \tag{59}$$

$$c_2(s) = \phi(s)Y_2(s) + (1-\alpha(s))Y_2(s) \tag{60}$$

$$e_2(s,i) = -\phi(s)Y_2(s) + \alpha(s) g_i Y_2(s) \tag{61}$$

$$e_2(s) = \mathbb{E}_i e_2(s,i) \tag{62}$$

$$\frac{c_1^{1-\gamma}}{1-\gamma} + \mathbb{E}_s \frac{c_2(s)^{1-\gamma}}{1-\gamma} \geq \underline{u} \tag{63}$$

for all (s,i) . Equations (58) and (59) are the resource constraints for periods one and two. Equations (60) and (61) pin down consumption for the consumer and the entrepreneur in period two. The last constraint maps the Pareto frontier. Lets re-write the program in terms of consumption of the entrepreneur:

$$\max_{\{e_1,\phi(s)\}_{s\in S}} \frac{e_1^{1-\gamma}}{1-\gamma} + \mathbb{E}_{i,s}\frac{\left(-\phi(s)Y_2(s) + \alpha(s) g_i Y_2(s)\right)^{1-\gamma}}{1-\gamma}$$

$$\frac{(Y_1 - e_1)^{1-\gamma}}{1-\gamma} + \mathbb{E}_s \frac{(\phi(s)Y_2(s) - \alpha(s)Y_2(s))^{1-\gamma}}{1-\gamma} \geq \underline{u}.$$

The first order conditions are

$$\begin{aligned}
e_1 &: e_1^{-\gamma} - \lambda\left(Y_1 - e_1\right)^{-\gamma} = 0 \\
\phi(s) &: -Y_2(s)\Pi(s)\mathbb{E}_i\left(-\phi(s)Y_2(s) + \alpha(s)g_iY_2(s)\right)^{-\gamma} \\
&\quad + \lambda Y_2(s)\Pi(s)\left(\phi(s)Y_2(s) + (1-\alpha(s))Y_2(s)\right)^{-\gamma} = 0
\end{aligned}$$

Thus, for every state the ratio of consumption is equal to:

$$\frac{e_1^{-\gamma}}{c_1^{-\gamma}} = \frac{\mathbb{E}_i\left(-\phi(s)Y_2(s) + \alpha(s)g_iY_2(s)\right)^{-\gamma}}{\left(\phi(s)Y_2(s) + (1-\alpha(s))Y_2(s)\right)^{-\gamma}}.$$

This is exactly the same relative allocation of consumption. Note that the level of consumption will depend on the reservation utility $\underline{u}$. Thus, for a particular choice of $\underline{u}$ we can recover the allocation of the competitive equilibrium.

Efficiency We summarize the discussion in the following proposition. A competitive equilibrium is Pareto Efficient if there exists some $\underline{u}^*$ such that the allocation of the competitive equilibrium for a given initial distribution of wealth (α_1, ϕ_1)[34] is equal to the solution of the planning problem for a level of reservation utility $\underline{u}^*$.

Proposition 4. *The competitive equilibrium with id risk is the solution of the planning problem when*

$$\underline{u} = \frac{\left(Y_1 - e_1^{IM}\left(\alpha_1, \phi_1\right)\right)^{1-\gamma}}{1-\gamma} + \mathbb{E}_s \frac{\left(\phi^{IM}(s)\left(\alpha_1, \phi_1\right)Y_2(s) - \alpha(s)Y_2(s)\right)^{1-\gamma}}{1-\gamma}. \tag{64}$$

Thus, the competitive equilibrium is constrained efficient.

The proof is immediate. For the level $\underline{u}$ defined in (64): the participation constraint of the consumer holds with equality, and the allocation of incomplete markets meets all of the first order conditions hold. In the case that the planner has more instruments, in particular, it can perfectly control consumption, then the planning problem will be the same one as the complete markets allocation. Thus, we need to make a choice regarding which problem we are focusing on. Note that this result is coming from the fact that in the two period problem we are not micro-founding the amount of id risk that the entrepreneurs

[34] Note that this pins down the initial assets of each one the consumer and the entrepreneur and initial income (that depends on the capital and labor share).

face. This is also the case in the infinite horizon. The source of the inefficiency, and some other papers in the literature is in there.

F General Model with Epstein-Zin's preferences.

In this appendix we present a details characterization of the equilibrium. These equations are support expressions presented in Section 3.

F.1 Worker's problem

Any representative worker solves:

$$V^w(a,s) = \max_{\{c(s),a(s'|s)\}} \left\{ \frac{(c(s))^{1-\sigma}}{1-\sigma} + \beta \left(\mathbb{E} V^w \left(a\left(s' \mid s\right), s'\right)^{\frac{1-\gamma}{1-\sigma}} \right)^{\frac{1-\sigma}{1-\gamma}} \right\}$$

$$c(s) + \sum_{s'} p\left(s' \mid s\right) a\left(s' \mid s\right) \leq a(s) + \omega(s)$$

We guess and verify that the solution has the following structure:

$$\begin{aligned} V(s) &= \left(A(s)(a+\omega(s)+h(s))\right)^{1-\sigma} \\ c(s) &= (1-\zeta(s))(a+\omega(s)+h(s)) \\ h(s) &= \sum_{s'|s} p\left(s' \mid s\right)\left[\omega\left(s'\right)+h\left(s'\right)\right] \\ a\left(s' \mid s\right) &= \phi^w\left(s' \mid s\right)\zeta(s)[a+\omega(s)+h(s)] - \omega\left(s'\right) - h\left(s'\right) \end{aligned}$$

The unknowns here are

$$\boxed{A(s), \zeta(s), \phi\left(s' \mid s\right)}$$

once we have these we can obtain $c(s), a\left(s' \mid s\right)$. Taking the FOC we have

$$\begin{aligned} p\left(s' \mid s\right)(c(s))^{-\sigma} =& \beta\left(\mathbb{E}\left[\left(A(s')(a\left(s' \mid s\right)+\omega(s' \mid s)+h(s' \mid s))\right)^{1-\gamma}\right]\right)^{\frac{1-\sigma}{1-\gamma}-1} \\ & \Pi(s' \mid s)\left[A(s')(a\left(s' \mid s\right)+\omega(s' \mid s)+h(s' \mid s))\right]^{-\gamma} A(s') \end{aligned}$$

Rename for easier notation $W^w(s) \equiv a+\omega(s)+h(s)$. Then we have:

$$p\left(s' \mid s\right)(c(s))^{-\sigma} = \beta\left(\mathbb{E}\left[\left(A(s')W^w(s')\right)^{1-\gamma}\right]\right)^{\frac{1-\sigma}{1-\gamma}-1} \Pi(s' \mid s)\left[A(s')W^w(s')\right]^{-\gamma} A(s') \tag{65}$$

From the latter equation we see that using the guessed solution we can find

$$\phi^w(s' \mid s)^\gamma = \left(\frac{\zeta(s)}{(1-\zeta(s))}\right)^{\frac{1-\gamma}{1-\frac{1}{\sigma}}} \tilde{\beta}(s',s)^\gamma (1-\zeta(s'))^{\frac{1-\gamma}{1-\frac{1}{\sigma}}}$$

Which gives us the workers' asset positions. Where $\tilde{\beta}(s',s) = \frac{\beta^{\frac{(1-\gamma)}{\gamma(1-\sigma)}} \Pi(s'|s)^{\frac{1}{\gamma}}}{p(s'|s)^{\frac{1}{\gamma}}}$. Note that with for CRRA preferences and $\sigma = \gamma$ the expression simplifies to:

$$\tilde{\beta}(s',s) = \hat{\beta}(s',s) = \left(\frac{\beta \Pi(s' \mid s)}{p\,(s' \mid s)}\right)^{\frac{1}{\gamma}}$$

Now we have to find an equation for the savings ratio. Multiply both sides of (65) by $W^w(s')$ and add up over s', then use the guesses for the law of motion of wealth and consumption, and after some algebra we get to:

$$\left(\frac{(1-\zeta(s))}{\zeta(s)}\right)^{-\sigma} \sum_{s'} p\,(s' \mid s)\,\phi^w(s' \mid s) = \beta \left(\mathbb{E}\left[\left(A(s')\phi^w(s' \mid s)\right)^{1-\gamma}\right]\right)^{\frac{1-\sigma}{1-\gamma}}$$

Now, recall that $\sum_{s'} p\,(s' \mid s)\,\phi^w(s' \mid s) = 1$. Thus, we have this useful relationship:

$$\left(\frac{(1-\zeta(s))}{\zeta(s)}\right)^{-\sigma} = \beta \left(\mathbb{E}\left[\left(A(s')\phi^w(s' \mid s)\right)^{1-\gamma}\right]\right)^{\frac{1-\sigma}{1-\gamma}} \tag{66}$$

We can use the relationship in equation (66) to simplify the FOC (65).

Now we use the envelope theorem to get a value for $A(s)$. Use again the guessed solution. We know that the envelope theorem states that

$$\begin{aligned}
V'(W^w(s)) &= U'(c(s)) \\
A(s)U'(A(s)W^w(s)) &= U'(c(s)) \\
A(s)(A(s)W^w(s))^{-\sigma} &= [(1-\zeta(s))W^w(s)]^{-\sigma} \\
A(s)^{1-\sigma} &= (1-\zeta(s))^{-\sigma}.
\end{aligned}$$

The last equation implies $A(s)^{1-\gamma} = (1-\zeta(s))^{\frac{1-\gamma}{1-\frac{1}{\sigma}}}$, so collecting terms and using this last relationship:

$$\left(\frac{(1-\zeta(s))}{\zeta(s)}\right)^{\frac{1-\gamma}{1-\frac{1}{\sigma}}} = \frac{\beta^{\frac{1-\gamma}{1-\sigma}} \Pi(s' \mid s)}{p\,(s' \mid s)} (1-\zeta(s'))^{\frac{1-\gamma}{1-\frac{1}{\sigma}}} \phi^w(s' \mid s)^{-\gamma} \tag{67}$$

As the objective is to solve for savings ratios, we would like to write a linear equation.

To do so, we multiply the whole expression by the price and sum over states to get to:

$$\frac{1}{\zeta(s)} = 1 + \left[\sum_{s'} p(s'|s)\tilde{\beta}(s',s)(1-\zeta(s'))^{\frac{1-\gamma}{\gamma(1-\frac{1}{\sigma})}}\right]^{\frac{\gamma(1-\frac{1}{\sigma})}{1-\gamma}}$$

F.2 Capitalists' problem

The program of the entrepreneur is given by

$$V^e(E,k;s,i) = \max_{\{e(s,i),E(s'|s),k'(s,i)\}} \left\{U(e(s,i) + \beta U\left(\mathbb{CE}\left(V^e\left(E\left(s' \mid s\right), k(s' \mid s; i' \mid i), s', i'\right)\right)\right)\right\}$$

$$e(s,i) + k'(s,i) + \sum_{s'} p\left(s' \mid s\right) E\left(s' \mid s\right) \leq E(s) + r(s)kg_i$$

where $U(x) = \frac{x^{1-\sigma}}{1-\sigma}$, $\mathbb{CE}(x) = \Gamma^{-1}\left[\mathbb{E}\Gamma U^{-1}(x)\right]$, $\Gamma(x) = \frac{x^{1-\gamma}}{1-\gamma}$ The guesses are

$$\begin{aligned} e(s,i) =& (1-\vartheta(s))W^e(s,i,k) \\ k'(s,i) =& \nu(s)\vartheta(s)W^e(s,i,k) \\ E(s' \mid s,i) =& \phi^e(s' \mid s)E_1(s,i) \end{aligned}$$

where $\vartheta(s)$ is the entrepreneur's saving rate, $\nu(s)$ is the portion of savings invested in capital and the total wealth is defined by:

$$W^e(s,i,k) = E(s,i) + r(s)g_ik.$$

We can also guess that

$$V^e(E,k;s,i) = U(B(s)W^e)$$

$$V^e(E,k;s,i) = \frac{(B(s)W^e)^{1-\sigma}}{1-\sigma}$$

where $B(s)$ is the marginal utility of wealth. Thus, the problem becomes:

$$\begin{aligned} V^e(E,k;s,i) = \max_{\{e(s,i),E(s'|s),k'(s,i)\}} & \left\{\frac{(e(s,i))^{1-\sigma}}{1-\sigma} + \frac{\beta}{1-\sigma}\left[\mathbb{E}_{s',i'}\left\{\left(B(s')(E(s' \mid s) + r(s' \mid s)g_i'k')\right)^{1-\gamma}\right\}\right]^{\frac{1-\sigma}{1-\gamma}}\right\} \\ s.t. \quad & e(s,i) = E(s) + r(s)kg_i - k'(s,i) - \sum_{s'} p\left(s' \mid s\right) E\left(s' \mid s\right) \end{aligned}$$

We will take FOC against $E(s' \mid s)$ and $k'()$. These are given by:

$$E(s' \mid s): \quad p(s'|s)c(s)^{-\sigma} = \beta \left[\mathbb{E}_{s',i}[(B(s')W^e(s',i))^{1-\gamma}]\right]^{\frac{\gamma-\sigma}{1-\gamma}} \\ \Pi(s'|s)B(s')^{1-\gamma}\mathbb{E}_i[W^e(s',i)^{-\gamma}]; \;\; \forall s' \tag{68}$$

By changing the amount of arrow securities, we diminish consumption today, but we increase (decrease) the payoff in the future in state s'. The right hand side is the marginal change in the continuation utility. Note that in the case of risk neutrality, the correction of idiosyncratic risk is equal to one. The first order condition for capital is given by

$$k'(.): \quad c(s)^{-\sigma} = \beta \left[\mathbb{E}_{s',i}[(B(s')W^e(s',i'))^{1-\gamma}]\right]^{\frac{\gamma-\sigma}{1-\gamma}} \\ \mathbb{E}_{s',i}[B(s')^{1-\gamma}W^e(s',i)^{-\gamma}r(s')g_i] \tag{69}$$

Define

$$W^e(s',i',k') = E(s') + r(s')g_i'k'.$$

$$o(s',i;\phi^e,v) = \phi^e(s' \mid s)(1-v(s)) + r(s')g_i'\nu(s)$$

Using the guesses stated before and that

$$\begin{aligned} E_1(s,i) &\equiv \sum_{s'} p\left(s' \mid s\right) E\left(s' \mid s\right) \\ &= E(s) + r(s)kg_i - k'(s,i) - e(s,i) \\ &= W^e - \nu(s)\vartheta(s)W^e - (1-\vartheta(s))W^e \\ &= \vartheta(s)(1-v(s))W^e(s,i,k) \end{aligned}$$

we can get a law of motion for individual wealth

$$W^e(s',i',k') = \vartheta(s)o(s',i;\phi^e,v)W^e(s,i,k) \tag{70}$$

This still holds even in the case of Epstein Zin preferences. Notice that, as in the worker's problem we can use the envelope theorem to obtain $B(s' \mid s)$:

$$\begin{aligned} V'(W^e(s,i,k)) =& U'(e(s)) \\ B(s)^{1-\gamma} =& (1-\vartheta(s))^{\frac{1-\gamma}{1-1/\sigma}} \end{aligned} \tag{71}$$

Also recall that $o(s',i;\phi^e,v) = \phi^e(s' \mid s)(1-\nu(s)) + r(s')g_i v(s)$. We want to take a

Taylor expansion of the latter around $\mathbb{E}g_i = 1$, it will be useful later.

$$\begin{aligned}\mathbb{E}_{i'}\left(\left[o(s',i';\phi^e,v)\right]^{-\gamma}\right) &= \mathbb{E}_{i'}\left(\left[\phi^e(s' \mid s)(1-\nu(s)) + r(s')g_i'\nu(s)\right]^{-\gamma}\right)\\ &= \left[\phi^e(s' \mid s)(1-\nu(s)) + r(s')\nu(s)\right]^{-\gamma}\left(1+\tilde{R}(s'|s)\right)\end{aligned}$$

where

$$\mathbb{R}(s',s) = \left[1 + \frac{\gamma(1+\gamma)\left(\nu(s)r\left(s'\right)\right)^2 Var\left(g_i\right)}{2o\left(s',1;\phi^e,\nu\right)^2}\right]. \tag{72}$$

Now, to find an equation for the risk premium, add up over s' equation (68) and equalize to (69) to get:

$$\mathbb{E}_{s',i}\left[B(s')^{1-\gamma}W^e(s',i)^{-\gamma}\left(r(s')g_i - \frac{1}{\sum_{s'} p(s'|s)}\right)\right] = 0$$

Use the law of motion of wealth:

$$\mathbb{E}_{s',i}\left[B(s')^{1-\gamma}o(s',i;\phi^e,v)^{-\gamma}\left(r(s')g_i - \frac{1}{\sum_{s'} p(s'|s)}\right)\right] = 0 \tag{73}$$

This equation determines the investment ratio $\nu(s)$. Recall that $B(s)^{1-\gamma} = (1-\vartheta(s))^{\frac{1-\gamma}{1-1/\sigma}}$ which replaced in the last generates equation (35) in the paper. We want to use the expansion we computed before to simplify this equation. After some algebra we get to:

$$\begin{aligned}&\sum_{s'|s}\Pi(s'|s)B(s')^{1-\gamma}o(s',1)^{-\gamma}r(s' \mid s)\left(\mathbb{R}(s',s) - \gamma\frac{\nu(s)r\left(s'\right)Var\left(g_i\right)}{o\left(s',1\right)}\right) =\\ &\qquad\qquad\sum_{s'|s}\Pi(s'|s)B(s')^{1-\gamma}o(s',1)^{-\gamma}\mathbb{R}(s',s)\end{aligned}$$

Also notice that:

$$\begin{aligned}&\left[\phi^e(s' \mid s)(1-\nu(s)) + r(s')\nu(s)\right]^{-\gamma}B(s')^{1-\gamma} =\\ &\qquad\left(\frac{(1-\vartheta(s))}{\vartheta(s)}\right)^{-\sigma}\frac{p(s'|s)}{\beta\Pi(s'|s)}\left[\mathbb{E}_{s',i'}[(B(s')o(s',i'))^{1-\gamma}]\right]^{-\frac{\gamma-\sigma}{1-\gamma}}\mathbb{R}(s',s)^{-1}\end{aligned} \tag{74}$$

Combining the two and simplifying we obtain:

$$\sum_{s'|s}p(s'|s)r(s'|s)\left(1 - \gamma\frac{\nu(s)r\left(s'\right)Var\left(g_i\right)}{o\left(s',1\right)\mathbb{R}(s',s)}\right) = 1 \tag{75}$$

Which is the equation for the risk premium in the case of Epstein Zin preferences.

Note that this implies a capitalist's growth rate of wealth since:

$$\sum_{s'|s} p(s'|s)r(s'|s) = 1 + \sum_{s'|s} p(s'|s)r(s'|s)\gamma \frac{\nu(s)r(s')\,\mathbb{V}ar(g_i)}{o(s',1)\,\mathbb{R}(s',s)}$$

The last can also be written as:

$$1-\nu(s)+\nu(s)\sum_{s'|s} p(s'|s)r(s' \mid s) = 1 + \sum_{s'|s} p(s'|s)\left(\gamma \frac{\nu(s)^2 r(s')^2\,\mathbb{V}ar(g_i)}{o(s',1)\,\mathbb{R}(s',s)}\right)$$

Where the last term is the risk premium. In particular the above relationship implies:

$$\sum_{s'|s} p(s'|s)o(s',1) = \sum_{s'|s} p(s'|s)\left(\phi^e(s' \mid s)(1-\nu(s)) + r(s')\nu(s)\right) = 1 + Prem(s) \tag{76}$$

where we used the fact that $\sum_{s'|s} p(s' \mid s)\phi^e(s' \mid s) = 1$. Having found an equation for the risk premium, we now want to find the equations for $\vartheta(s), \phi^e(s' \mid s)$. First we need an additional Taylor expansion. expand the term $\mathbb{E}_{i'}\left([o(s',i;\phi^e,\nu)]^{1-\gamma}\right)$:

$$\mathbb{E}_{i'}\left(\left[o(s',i)\right]^{1-\gamma}\right) = \left[\phi^e(s' \mid s)(1-\nu(s)) + r(s')\nu(s)\right]^{1-\gamma} R_1(s',s)$$

where:

$$R_1(s',s) = 1 - \gamma(1-\gamma)\frac{(\nu(s)r(s'))^2}{2o(s',1)^2}\mathbb{V}ar(g_i)$$

where $o(s',1) = \phi^e(s' \mid s)(1-\nu(s)) + r(s')\nu(s)$. Replacing the last in (74), reorganising, multiplying both sides by $p(s'|s)o(s',1)R_1(s',s)$, adding over s', we obtain:

$$\beta\left[\mathbb{E}_{s'}[\left[B(s')o(s',1)\right]^{1-\gamma} R_1(s',s)]\right]^{\frac{1-\sigma}{1-\gamma}} = \left(\frac{(1-\vartheta(s))}{\vartheta(s)}\right)^{-\sigma}\left[1+\tilde{Prem}(s)\right] \tag{77}$$

note that $\tilde{Prem}(s) = 0$, as we can write :

$$\frac{R_1(s',s)}{\mathbb{R}(s',s)} = 1 - \gamma\frac{(\nu(s)r(s'))^2}{o(s',1)^2}\frac{\mathbb{V}ar(g_i)}{\mathbb{R}(s',s)}$$

and define:

$$\tilde{Prem}(s) = Prem(s) - \gamma Var(g_i)\sum_{s'}\frac{p(s'|s)}{\mathbb{R}(s',s)}\frac{(\nu(s)r(s'))^2}{o(s',1)}$$

Since $Prem(s)$ is exactly equal to the last term we have $\tilde{Prem}(s) = 0$. But using equation

(77) we can find an equation for the savings ratio ϑ. After some algebra we get

$$(1-\vartheta(s'))^{\frac{1-\gamma}{1-1/\sigma}}\frac{\beta^{\frac{(1-\gamma)}{(1-\sigma)}}\Pi(s'|s)}{p(s'|s)}R(s',s)=o(s',1)^{\gamma}\left(\frac{(1-\vartheta(s))}{\vartheta(s)}\right)^{\frac{1-\gamma}{1-1/\sigma}} \tag{78}$$

Multiplying by $p(s'|s)$, using the fact that $\sum_{s'|s}p(s'|s)o(s',1)=1+Premia(s)$, the definition of $\tilde{\beta}(s',s)$, equation (76) adding up over s' we get

$$\sum_{s'|s}p(s'|s)\tilde{\beta}(s',s)(1-\vartheta(s'))^{\frac{1-\gamma}{\gamma(1-1/\sigma)}}\mathbb{R}(s'|s)^{\frac{1}{\gamma}}=(1+Prem(s))\left(\frac{(1-\vartheta(s))}{\vartheta(s)}\right)^{\frac{1-\gamma}{\gamma(1-1/\sigma)}}$$

Defining now the "ratio":

$$Ra(s',s)=\frac{\mathbb{R}(s'|s)^{\frac{1}{\gamma}}}{(1+Prem(s))}$$

we can rewrite

$$\left(\frac{(1-\vartheta(s))}{\vartheta(s)}\right)^{\frac{1-\gamma}{\gamma(1-1/\sigma)}}=\sum_{s'|s}p(s'|s)\tilde{\beta}(s',s)(1-\vartheta(s'))^{\frac{1-\gamma}{\gamma(1-1/\sigma)}}Ra(s',s) \tag{79}$$

Which has the same form as with CRRA preferences and it is easy to see that it is the same as in the draft when $\gamma=\sigma$. We can rewrite the latter equation as

$$\frac{1}{\vartheta(s)}=1+\left[\sum_{s'}p(s'|s)\tilde{\beta}(s',s)(1-\vartheta(s'))^{\frac{1-\gamma}{\gamma(1-1/\sigma)}}Ra(s',s)\right]^{\frac{\gamma(1-1/\sigma)}{1-\gamma}}$$

To recover ϕ^e we use equation (78) writing it as:

$$(1-\nu(s))\phi^e(s',s)=\left(\frac{\vartheta(s)(1-\vartheta(s'))}{(1-\vartheta(s))}\right)^{\frac{1-\gamma}{\gamma(1-1/\sigma)}}\tilde{\beta}(s',s)\mathbb{R}(s'|s)^{\frac{1}{\gamma}}-\nu(s)r(s') \tag{80}$$

F.3 Equilibrium and prices

Now **assets market** clearing reads:

$$\phi^w(s'|s)\zeta(s)x+\phi^e(s'|s)[\vartheta(s)(1-\nu(s))](1-x)=\frac{\omega(s')+h(s')}{W^T(s)};\ \forall s,s' \tag{81}$$

We also have the **goods market clearing** to check:

$$\begin{aligned} c(s)+e(s)+k'(s) &= y(s);\ \ \forall s \\ (1-\zeta(s))W^c(s)+(1-\vartheta(s))W^e(s)+\vartheta(s)\nu(s)W^e(s) &= y(s);\ \ \forall s \\ (1-\zeta(s))x+[1-\vartheta(s)(1-\nu(s))](1-x) &= \tfrac{y(s)}{W^T(s)};\ \forall s \end{aligned} \tag{82}$$

We use the asset market clearing to find the prices. Recall that the equation for ϕ^e is:

$$\phi^e(s',s)=\left(\frac{\vartheta(s)(1-\vartheta(s'))}{(1-\vartheta(s))}\right)^{\frac{1-\gamma}{\gamma(1-1/\sigma)}}\tilde{\beta}^e(s',s)\frac{\mathbb{R}(s',s)^{\frac{1}{\gamma}}}{(1-\nu(s))}-\frac{\nu(s)r(s')}{(1-\nu(s))}$$

where $\tilde{\beta}^e(s',s)=\frac{(\beta^e)^{\frac{(1-\gamma)}{\gamma(1-\sigma)}}\Pi(s'|s)^{\frac{1}{\gamma}}}{p(s'|s)^{\frac{1}{\gamma}}}$.

The equivalent expression for the worker is given by:

$$\phi^w(s'\mid s)=\left(\frac{\zeta(s)(1-\zeta(s'))}{(1-\zeta(s))}\right)^{\frac{1-\gamma}{\gamma(1-1/\sigma)}}\tilde{\beta}(s',s)$$

where $\tilde{\beta}(s',s)=\frac{\beta^{\frac{(1-\gamma)}{\gamma(1-\sigma)}}\Pi(s'|s)^{\frac{1}{\gamma}}}{p(s'|s)^{\frac{1}{\gamma}}}$.

Using both in the asset market clearing

$$\phi^w(s'\mid s)\zeta(s)x+\phi^e(s'\mid s)\vartheta(s)(1-\nu(s))(1-x)=\frac{\omega(s')+h(s')}{T(s)}\equiv M(s',s)$$

We have the equation for prices

$$\frac{p(s'|s)}{\beta^{\frac{(1-\gamma)}{(1-\sigma)}}\Pi(s'|s)}=\left\{\frac{\left(\frac{\zeta(s)(1-\zeta(s'))}{(1-\zeta(s))}\right)^{\frac{1-\gamma}{\gamma(1-1/\sigma)}}\zeta(s)x+\left(\frac{\beta^e}{\beta}\right)^{\frac{(1-\gamma)}{\gamma(1-\sigma)}}\left(\frac{\vartheta(s)(1-\vartheta(s'))}{(1-\vartheta(s))}\right)^{\frac{1-\gamma}{\gamma(1-1/\sigma)}}\mathbb{R}(s'|s)^{\frac{1}{\gamma}}\vartheta(s)(1-x)}{r(s')\nu(s)\vartheta(s)(1-x)+M(s',s)}\right\}^{\gamma} \tag{83}$$

Note also that:

$$r(s')\nu(s)\vartheta(s)(1-x)+M(s',s)=\frac{W^T(s')}{W^T(s')}$$

which leads to the expression in Section 3.3.

G Risk premium correlation with GDP

In this section we compute the correlations, including transition paths, when the economy starts from a very low wealth ratio x. To be precise, we start with $x_0 = 0.37$. The next table shows that the average moments are similar to the main calibration in Section 4.2. Thus, the slight differences with that section are due to the inclusion of the transition path in the computation of the moments. For this computation we simulate for only 1500 periods. This "short," but not excessively so, sample allows us to approximate the average moments in Table 2, while simultaneously maximize the impact of the transition path.

Table 5: Averages including transition paths.

δ	α_k	$\mathbb{E}(\alpha(s))$	$\mathbb{V}ar(\alpha(s))$	$\frac{K}{Y}$	$\mathbb{E}(x)$	A	B	σ	γ	Premium
0.07	0.265	0.374	0.0067	2.812	0.815	0.776	-1.607	2.0	5.0	0.0527

Table 6 shows the observed correlations. Notice that correlation between $\alpha(s)$ and the risk premium is reduced from 0.92 in Table 3 to 0.02 in this exercise. Since there is a perfect positive correlation between α and GDP (by construction), this also implies a reduced correlation of the risk premium with GPD. What this exercise is doing is maximizing the impact of the wealth effects described in Section 3.5. Because the starting x is significantly low and distant from the stationary one ($x_0 = 0.37$ vs $E(x_t) = 0.83$), during the transition x_t is mostly increasing. Notice, though, that speed of convergency is notably slow: even with 1500 periods the effect of the transition path have an impact on the average. If we were to reduce the sample to 100 periods, the correlation would definitively be negative, but then the averages would be off.

Table 6: Implied correlations with low x_0.

α_t	B_t	A_t	R_t	Risk Premium
1.00	-0.99	-0.22	-0.88	0.02
-0.99	1.00	0.15	0.90	-0.08
-0.22	0.15	1.00	-0.28	0.97
-0.88	0.90	-0.28	1.00	-0.50
0.02	-0.08	0.97	-0.50	1.00

H Proof of Proposition 3

H.1 Proof of Part (a)

First notice that when capitalists can fully insure their idiosyncratic risk $m(s) = 1$, for all s. Then from equations (25) and (33) follows that $\zeta(s) = \vartheta(s)$. The wealth ratio's law of motion satisfies is given by $x(s'|s) = W^w(s'|s)/W^T(s'|S)$. Using the laws of motion of each agents wealths it can be written as:

$$x(s'|s) = \frac{\phi^w(s'|s)\zeta(s)x}{\mathbb{E}_i o(s', i, s)\vartheta(s)(1-x) + \phi^w(s'|s)\zeta(s)x}$$

where from equation (34), $\mathbb{E}_i o(s', i; \phi^e)^{-\sigma}$ satisfies:

$$\left(\mathbb{E}_i o(s', i; \phi^e)^{-\sigma}\right)^{-\frac{1}{\sigma}} = \tilde{\beta}(s', s)\frac{(1-\vartheta(s))}{\vartheta(s)(1-\vartheta(s'))}; \;\; \forall s, s'$$

When all idiosyncratic risk is insured $\mathbb{E}_i o(s', i; \phi^e)^{-\sigma} = [\mathbb{E}_i o(s', i; \phi^e)]^{-\sigma}$, thus using this and $\zeta(s) = \vartheta(s)$, from the last equation and (24) we obtain:

$$\mathbb{E}_i o(s', i; \phi^e) = \frac{\tilde{\beta}(s', s)(1-\vartheta(s))}{\vartheta(s)(1-\vartheta(s'))} = \frac{\tilde{\beta}(s', s)(1-\zeta(s))}{\zeta(s)(1-\zeta(s'))} = \phi^w(s'|s); \;\; \forall s, s'$$

Using the last equation in the first delivers the result.

H.2 Proof of Part (b)

Assume that $\delta = 1$ and guess an equilibrium with constant x. To do this, assume that capitalists have a different discount rate β^e. We will pick its value to make sure that x is constant. Since we are assuming $\gamma = \sigma$, we use a guess and verify strategy, guessing that prices and the risk adjustment factor satisfy:

$$p(s'|s) = A_0\beta\Pi(s'|s)\tilde{g}(s'|s)^{-\sigma} \qquad \textit{and} \qquad m(s) = m; \qquad \forall s, s' \tag{84}$$

for some constant $A_0 > 0$ and $m \geq 1$; $\forall x$. Later we verify this guess. We prove this proposition in a series of steps showing that: 1) the savings rates are independent of the state, 2) holdings of contingent assets are proportional to growth, 3) the investment rate and portfolio allocations are constant, 4) the wealth growth rates are independent of the state. In the final steps we verify the guesses in (84).

Savings rates are independent of aggregate shock. Using the guessed prices into the definition of $\tilde{\beta}(s'|s)$ we obtain:

$$\tilde{\beta}(s'|s) = \left[\frac{\beta\Pi(s'|s)}{p(s'|s)}\right]^{1/\sigma} = \left[\frac{\beta\Pi(s'|s)}{A_0\beta\Pi(s'|s)\tilde{g}(s'|s)^{-\sigma}}\right]^{1/\sigma} = \frac{\tilde{g}(s'|s)}{A_0^{1/\sigma}}$$

And:

$$\tilde{\beta^e}(s'|s) = \left[\frac{\beta^e\Pi(s'|s)}{p(s'|s)}\right]^{1/\sigma} = \left[\frac{\beta^e\Pi(s'|s)}{A_0\beta\Pi(s'|s)\tilde{g}(s'|s)^{-\sigma}}\right]^{1/\sigma} = \left(\frac{\beta^e}{\beta}\right)^{1/\sigma}\frac{\tilde{g}(s'|s)}{A_0^{1/\sigma}}$$

Guess that the savings rates are constant. The last in equation (25) together with $\gamma = \sigma$ and the guessed price imply that the solution for the worker's saving rate is:

$$\zeta(s') = \zeta(s) = \zeta = \beta\frac{\mathbb{E}\tilde{g}(s'|s)^{1-\sigma}}{A_0^{\frac{1-\sigma}{\sigma}}}; \qquad \forall s, s'$$

Doing similar calculations with equation (33) we obtain:

$$\vartheta(s) = \vartheta = \beta\frac{\mathbb{E}\tilde{g}(s'|s)^{1-\sigma}}{mA_0^{\frac{1-\sigma}{\sigma}}}\left(\frac{\beta^e}{\beta}\right)^{1/\sigma}; \qquad \forall s, s' \tag{85}$$

AD securities are proportional to $\tilde{g}(s'|s)$. From equation (24) with $\gamma = \sigma$ and the computed value of $\tilde{\beta}(s'|s)$ we obtain:

$$\phi^w(s'|s) = \frac{\tilde{g}(s'|s)}{\zeta A_0^{1/\sigma}} = \frac{\tilde{g}(s'|s)}{\beta A_0\mathbb{E}\tilde{g}(s'|s)^{1-\sigma}}.$$

The equivalent condition for the capitalist (see equation (80) in the online appendix) generates:

$$\left[\mathbb{E}_i o(s', i, \phi^e)^{-\sigma}\right]^{-1/\sigma} = \left(\frac{\beta^e}{\beta}\right)^{1/\sigma}\frac{\tilde{g}(s'|s)}{\vartheta A_0^{1/\sigma}} = \frac{\tilde{g}(s'|s)m}{\beta A_0\mathbb{E}\tilde{g}(s'|s)^{1-\sigma}} \qquad \forall s, s' \tag{86}$$

Note that this implies that $\phi^w(s'|s) = \frac{\tilde{\beta}(s'|s)}{\sum_{s'} p(s'|s)\tilde{\beta}(s'|s)}$ and $[\mathbb{E}_i o(s', i, \phi^e)^{-\sigma}]^{-1/\sigma} = \frac{\tilde{\beta}^e(s'|s)m}{\sum_{s'} p(s'|s)\tilde{\beta}(s'|s)}$

The capitalist's growth rate of wealth is $o(s', i; \phi^e) \equiv [(1 - \nu(s))\phi^e(s'|s) + \nu(s)r(s')g_i]$. Define the portfolio allocation as:

$$\frac{(1 - \nu(s))\phi^e(s'|s)}{\nu(s)r(s')} = \frac{1}{D}$$

Also guessing that D is constant. Then it is immediate (recall $\sum_{s'} p(s'|s)\phi^e(s'|s) = 1$) that

$$m(s) \equiv \sum_{s'|s} p(s'|s) \left(\mathbb{E}_i o(s',i;\phi^e)^{-\sigma}\right)^{-1/\sigma} = (1-\nu(s)) \left[\mathbb{E}_i (1+Dg_i)^{-\sigma}\right]^{-1/\sigma} \tag{87}$$

As a result:

$$\left[\mathbb{E}_i o(s',i,\phi^e)^{-\sigma}\right]^{-1/\sigma} = (1-\nu(s))\phi^e(s'|s) \left[\mathbb{E}_i (1+Dg_i)^{-\sigma}\right]^{-1/\sigma} = \frac{\tilde{\beta}(s'|s)m(s)}{\sum_{s'} p(s'|s)\tilde{\beta}(s'|s)}$$

Because of (87) the last implies $\phi^e(s'|s) = \phi^w(s'|s), \quad \forall s, s'$

Investment rates are constant. Using portfolio choice D in equation (35) :

$$\mathbb{E}_{s',i|s}\left[\left[(1-\nu(s))\phi^e(s'|s)(1+Dg_i)\right]^{-\sigma}\left(\frac{(1-\nu(s))\phi^e(s'|s)}{\nu(s)}Dg - \frac{1}{\beta A_0 \mathbb{E}\tilde{g}(s'|s)^{-\sigma}}\right)\right] = 0$$

Since i is independent of s we can write:

$$\mathbb{E}_{s'|s}\left[\frac{\left[(1-\nu(s))\phi^e(s'|s)\right]^{1-\sigma}}{\nu(s)}\right]\mathbb{E}_i\left((1+Dg_i)^{-\sigma}Dg_i\right) = \frac{\mathbb{E}_{s'|s}\left[(1-\nu(s))\phi^e(s'|s)\right]^{-\sigma}\mathbb{E}_i(1+Dg_i)^{-\sigma}}{\beta A_0 \mathbb{E}\tilde{g}(s'|s)^{-\sigma}}$$

$$\frac{(1-\nu(s))}{\nu(s)}\mathbb{E}_{s'|s}[\phi^e(s'|s)]^{1-\sigma}\mathbb{E}_i\left((1+Dg_i)^{-\sigma}Dg_i\right) = \frac{\mathbb{E}_{s'|s}\left[\phi^e(s'|s)\right]^{-\sigma}\mathbb{E}_i(1+Dg_i)^{-\sigma}}{\beta A_0 \mathbb{E}_{s'}\tilde{g}(s'|s)^{-\sigma}}$$

Doing similar calculations as before to replace $\phi^e(s'|s)$ we obtain:

$$\frac{\mathbb{E}_i\left((1+Dg_i)^{-\sigma}Dg_i\right)}{\mathbb{E}_i(1+Dg_i)^{-\sigma}} = \frac{\nu}{(1-\nu)} \tag{88}$$

So ν is constant whenever D is constant. Also, with D and ν we can compute the value of $m(s)$ given by equation (87), which confirms that m is constant. To solve for D use portfolio equation to get:

$$\frac{(1-\nu)}{\nu}D = \frac{r(s')}{\phi^e(s'|s)} = \frac{r(s')}{\phi^c(s'|s)} = \frac{r(s')}{\tilde{g}(s'|s)}\beta A_0 \mathbb{E}_{s'|s}\tilde{g}(s'|s)^{1-\sigma}$$

With an AK model $r(s')$ is exogenous, so the above equation pins down D, which it will not depend on x. In a more general setting $r(s')$ would depend on aggregate capital. With constant shares $r(s') = \alpha\frac{y(s')}{K'}$, which can be written as $r(s') = \alpha\tilde{g}(s'|s)\frac{y(s)}{K'}$. But since the capital law of motion is $K' = \vartheta\nu(1-x)W^T(s)$, we can write the previous equation as:

$$r(s') = \frac{\tilde{g}(s'|s)}{\beta A_0 \mathbb{E}_{s'|s}\tilde{g}(s'|s)^{1-\sigma}} \frac{(1-\nu)}{\nu} D$$

$$\alpha \tilde{g}(s'|s) \frac{y(s)}{\vartheta \nu (1-x) W^T(s)} = \frac{\tilde{g}(s'|s)}{\beta A_0 \mathbb{E}_{s'|s}\tilde{g}(s'|s)^{1-\sigma}} \frac{(1-\nu)}{\nu} D$$

$$\frac{\alpha}{\vartheta(1-x)} \frac{y(s)}{W^T(s)} = \frac{(1-\nu)D}{\beta A_0 \mathbb{E}_{s'|s}\tilde{g}(s'|s)^{1-\sigma}}$$

Now replacing ϑ from the previously found value:

$$\frac{\alpha m A_0^{\frac{1}{\sigma}}}{(1-x)} \frac{y(s)}{W^T(s)} = (1-\nu) D \left(\frac{\beta^e}{\beta}\right)^{1/\sigma}$$

$$\frac{\alpha \left[\mathbb{E}_i(1+Dg_i)^{-\sigma}\right]^{-1/\sigma} A_0^{\frac{1}{\sigma}}}{(1-x)} \frac{y(s)}{W^T(s)} = D \left(\frac{\beta^e}{\beta}\right)^{1/\sigma} \tag{89}$$

Where in the last step we have replaced m from equation (87). This equation solves for D, then (88) delivers ν, and then with (87) we obtain m. All this variables are constant if 1) α is constant and 2) the ratio $\frac{y(s)}{W^T(s)}$ is constant. What we need in general is that this solution is independent of the aggregate shock, it could depend on x or k as long as it does it in deterministic way. Here we are considering the case of x constant to simplify the calculations. We consider these cases in the following extensions of this proposition.

GDP-wealth ratio is constant. The human capital $h(s)$, can be written per unit of output. Defining $\tilde{h}(s) = \frac{h(s)}{Y(s)}$ it follows that:

$$\tilde{h}(s) = \sum_{s'|s} p(s'|s) \left([1-\alpha(s') + \tilde{h}(s')] \tilde{g}(s'|s) \right)$$

where $\tilde{g}(s'|s) = \frac{Y(s')}{Y(s)}$. Using the last, we can write:

$$\frac{w(s') + h(s')}{W^T(s)} = \tilde{g}(s'|s) \frac{[1-\alpha(s') + \tilde{h}(s')]}{R(s)(K/Y) + (1-\alpha(s)) + \tilde{h}(s)} \tag{90}$$

Suppose $\delta = 1$ Note that (84) implies that $\sum_{s'|s} p(s'|s) = \beta A_0 \mathbb{E}\tilde{g}(s'|s)^{-\sigma} = \tilde{p}$. Using equation (90) with constant shares we obtain:

$$\frac{w(s') + h(s')}{W^T(s)} = \tilde{g}(s'|s) \frac{[1-\alpha(s') + \tilde{h}(s')]}{1+\tilde{h}(s)} = \tilde{g}(s'|s) \frac{(1-\alpha)[1+G(s')]}{1+(1-\alpha)G(s)}$$

where $G(s) = \sum_{s'|s} p(s'|s) \left[(1+G(s'))\tilde{g}(s'|s)\right]$ is the present value of a constant dividend unit with grow factor $\tilde{g}(s'|s)$ Similarly:

$$\frac{W^T(s')}{W^T(s)} = \tilde{g}(s'|s)\frac{1+(1-\alpha)G(s')}{1+(1-\alpha)G(s)}$$

Because the distributions of growth rates are independent of the state and using the pricing function, it is straightforward to show that $G(s') = G(s) = G, \ \forall s, s'$; then it follows that:

$$\frac{W^T(s')}{W^T(s)} = \tilde{g}(s'|s); \qquad \frac{w(s')+h(s')}{W^T(s)} = \tilde{g}(s'|s)\frac{(1-\alpha)(1+G)}{1+(1-\alpha)G}.$$

Also,

$$\frac{y(s)}{W^T(s)} = \frac{y(s)}{y(s)+h(s')} = \frac{1}{1+\tilde{h}(s)} = \frac{1}{1+(1-\alpha)G}$$

Thus the ratio $\frac{y(s)}{W^T(s)}$ is constant and therefore also ν, D and m are.

Use feasibility to find A_0. We can use feasibility to dig further into the solutions. Replacing (87) in (85) generates:

$$\vartheta(1-\nu) = \beta\frac{\mathbb{E}\tilde{g}(s'|s)^{1-\sigma}}{A_0^{\frac{1-\sigma}{\sigma}}}\left(\frac{\beta^e}{\beta}\right)^{1/\sigma}\left[\mathbb{E}_i(1+Dg_i)^{-\sigma}\right]^{1/\sigma}$$

We can use the last equation in the feasibility constraint:

$$\zeta(x)x + \vartheta(s)(1-\nu(s))(1-x) = 1 - \frac{y(s)}{W^T(s)}; \ \forall s$$

$$\beta\frac{\mathbb{E}\tilde{g}(s'|s)^{1-\sigma}}{A_0^{\frac{1-\sigma}{\sigma}}}\left[x + (1-x)\left[\mathbb{E}_i(1+Dg_i)^{-\sigma}\right]^{1/\sigma}\left(\frac{\beta^e}{\beta}\right)^{1/\sigma}\right] = \frac{(1-\alpha)G}{1+(1-\alpha)G}; \ \forall s \tag{91}$$

Computing the present values with the guessed function for $p(s'|s)$ we obtain:

$$G = \frac{\beta A_0\mathbb{E}\tilde{g}(s'|s)^{1-\sigma}}{1-\beta A_0\mathbb{E}\tilde{g}(s'|s)^{1-\sigma}} \tag{92}$$

Hence, equation (91) generates the value of A_0.

Verify guessed prices. For the AD prices we compute a price equation akin to that in the online Appendix F (see equation (83) when $\gamma = \sigma$). To make the proof self-contained we replicate some calculations adapted to this environment. Recall that the AD securities must satisfy:

$$\phi^w(s'|s) = \left[\frac{\beta\Pi(s'|s)}{p(s'|s)}\right]^{1/\sigma} \frac{(1-\zeta(s))}{(1-\zeta(s'))\zeta(s)}; \quad \left[\mathbb{E}_i o(s', i, \phi^e)^{-\sigma}\right]^{-1/\sigma} = \left[\frac{\beta^e\Pi(s'|s)}{p(s'|s)}\right]^{1/\sigma} \frac{(1-\vartheta(s))}{(1-\vartheta(s'))\vartheta(s)}, \forall s, s'$$

Using the definition of D, the second equality can be written as:

$$(1-\nu)\phi^e(s'|s)\left[\mathbb{E}_i(1+Dg_i)^{-\sigma}\right]^{-1/\sigma} = \left[\frac{\beta^e\Pi(s'|s)}{p(s'|s)}\right]^{1/\sigma} \frac{(1-\vartheta(s))}{(1-\vartheta(s'))\vartheta(s)}, \forall s, s'$$

Replacing the last two in the assets' market clearing we obtain:

$$\left[\frac{\beta\Pi(s'|s)}{p(s'|s)}\right]^{1/\sigma}\left[x + (1-x)\left[\mathbb{E}_i(1+Dg_i)^{-\sigma}\right]^{1/\sigma}\left(\frac{\beta^e}{\beta}\right)^{1/\sigma}\right] = \tilde{g}(s'|s)\frac{(1-\alpha)(1+G)}{1+(1-\alpha)G}; \quad \forall s, s'$$

Using equation (91) the above becomes:

$$\left[\frac{\beta\Pi(s'|s)}{p(s'|s)}\right]^{1/\sigma}\left[\frac{(1-\alpha)G}{1+(1-\alpha)G}A_0^{\frac{1-\sigma}{\sigma}}\right] = \beta\mathbb{E}\tilde{g}(s'|s)^{1-\sigma}\tilde{g}(s'|s)\frac{(1-\alpha)(1+G)}{1+(1-\alpha)G}; \quad \forall s, s'$$

Therefore,

$$p(s'|s) = \beta\Pi(s'|s)\tilde{g}(s'|s)^{-\sigma}\left(\beta\frac{(1+G)}{A_0^{\frac{1-\sigma}{\sigma}}G}\mathbb{E}\tilde{g}(s'|s)^{1-\sigma}\right)^{-\sigma}$$

Using the solution for G from (92), we obtain the initially guessed price function $p(s'|s) = \beta\Pi(s'|s)\tilde{g}(s'|s)^{-\sigma}A_0$.

Verify that x is constant with the appropriate choice of β^e. To solve for the evolution of x, recall that

$$x(s'|s) = \frac{\phi^w(s'|s)\zeta(s)x}{\mathbb{E}_i o(s', i, s)\vartheta(s)(1-x) + \phi^w(s'|s)\zeta(s)x}$$

Using definition of $o(s', i, s)$ and because we already showed that $\phi^e(s', s) = \phi^w(s', s)$ we can write the last as:

$$x' = \frac{\zeta x}{[(1-\nu)(1+D)]\vartheta(1-x) + \zeta x}$$

Now, note that savings rates satisfy

$$\frac{\vartheta}{\zeta} = \left(\frac{\beta^e}{\beta}\right)^{1/\sigma}\frac{1}{m} = \left(\frac{\beta^e}{\beta}\right)^{1/\sigma}\frac{\left[\mathbb{E}_i(1+Dg_i)^{-\sigma}\right]^{1/\sigma}}{1-\nu}$$

where in the last step we have used (87). Thus, the last in the law of motion of x generates:

$$x' = \frac{x}{\left(\frac{\beta^e}{\beta}\right)^{1/\sigma}(1+D)\left[\mathbb{E}_i(1+Dg_i)^{-\sigma}\right]^{1/\sigma}(1-x)+x}$$

Which implies that for $x' = x$ to be true β^e must satisfy:

$$\beta^e = \beta\frac{(1+D)^{-\sigma}}{\mathbb{E}_i(1+Dg_i)^{-\sigma}} \tag{93}$$

Because of he Jensen's inequality and the convexity of the marginal utility, so that $\mathbb{E}_i(1+Dg_i)^{-\sigma} \geq (1+D)^{-\sigma}$, it is clear that $\beta^e < \beta$. Capitalists must have a smaller discount factor, otherwise x would converge to zero. The correction in the discount factor corrects the upwards drift in the capitalist's savings needs. When there is not exposure to idiosyncratic risk, there is no need for the correction.

Check solution is correct. Alternatively, the x's law of motion is characterized by:

$$x(s'|s) = \phi^w(s'|s)\zeta(s)\frac{W^T(s)}{W^T(s')}x.$$

Replacing the relationships for $\phi^w(s'|s)$, $\zeta(s)$ and the grow rate of wealth implies:

$$x(s'|s) = \frac{\tilde{g}(s'|s)}{A_0^{1/\sigma}}\frac{1}{\tilde{g}(s'|s)}x$$

Hence, if $x' = x$, it must be that $A_0 = 1$. Is this true? The value of A_0 is determined by equation (91), which can be written as:

$$\beta A_0\frac{\mathbb{E}\tilde{g}(s'|s)^{1-\sigma}}{A_0^{\frac{1}{\sigma}}}\left[x+(1-x)\left[\mathbb{E}_i(1+Dg_i)^{-\sigma}\right]^{1/\sigma}\left(\frac{\beta^e}{\beta}\right)^{1/\sigma}\right] = 1-\frac{y}{W^T}$$

To show that indeed $A_0 = 1$ first notice that equation (93) implies

$$\left[\mathbb{E}_i(1+Dg_i)^{-\sigma}\right]^{1/\sigma}\left(\frac{\beta^e}{\beta}\right)^{1/\sigma} = 1-D\left[\mathbb{E}_i(1+Dg_i)^{-\sigma}\right]^{1/\sigma}\left(\frac{\beta^e}{\beta}\right)^{1/\sigma}$$

Therefore, replacing the latter in the former:

$$\beta A_0\frac{\mathbb{E}\tilde{g}(s'|s)^{1-\sigma}}{A_0^{\frac{1}{\sigma}}}\left[1-(1-x)D\left[\mathbb{E}_i(1+Dg_i)^{-\sigma}\right]^{1/\sigma}\left(\frac{\beta^e}{\beta}\right)^{1/\sigma}\right] = 1-\frac{y}{W^T}$$

Now, collecting the term $(1-x)D\left[\mathbb{E}_i(1+Dg_i)^{-\sigma}\right]^{1/\sigma}\left(\frac{\beta^e}{\beta}\right)^{1/\sigma}$ in equation (89) and replacing in the above we obtain:

$$\beta A_0 \frac{\mathbb{E}\tilde{g}(s'|s)^{1-\sigma}}{A_0^{\frac{1}{\sigma}}}\left[1-\alpha A_0^{\frac{1}{\sigma}}\frac{y}{W^T}\right] = 1-\frac{y}{W^T}$$

Therefore we have:

$$\beta A_0 \mathbb{E}\tilde{g}(s'|s)^{1-\sigma}\left[\frac{1}{A_0^{\frac{1}{\sigma}}}+\frac{(1-\alpha\beta A_0\mathbb{E}\tilde{g}(s'|s)^{1-\sigma})}{\beta A_0\mathbb{E}\tilde{g}(s'|s)^{1-\sigma}}\frac{y}{W^T}\right]=1$$

We showed before that $\frac{y}{W^T}=\frac{1}{1+(1-\alpha)G}$, which using equation (92) generates

$$\frac{y}{W^T}=\frac{1-\beta A_0\mathbb{E}\tilde{g}(s'|s)^{1-\sigma}}{1-\alpha\beta A_0\mathbb{E}\tilde{g}(s'|s)^{1-\sigma}}$$

Replacing the latter in the former:

$$\beta A_0 \mathbb{E}\tilde{g}(s'|s)^{1-\sigma}\left[\frac{1}{A_0^{\frac{1}{\sigma}}}+\frac{1-\beta A_0\mathbb{E}\tilde{g}(s'|s)^{1-\sigma}}{\beta A_0\mathbb{E}\tilde{g}(s'|s)^{1-\sigma}}\right]=1$$

Which can only be true if $A_0 = 1$.