Robust Predictions in Dynamic Policy Games*

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Abstract

Dynamic policy games feature a wide range of equilibria. This paper provides a methodology for obtaining robust predictions. We focus on a model of sovereign debt, although our methodology applies to other settings, such as models of monetary policy or capital taxation. Our main result is a characterization of distributions over outcomes that are consistent with a subgame perfect equilibrium conditional on the observed history. We illustrate our main result by computing, conditional on an observed history, bounds across all equilibria on: the maximum probability of a crisis, means, variances, and covariances over debt prices.

Keywords: multiple equilibria, robustness, moment inequalities, correlated equilibrium, policy games.

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1 Introduction

Following Kydland and Prescott (1977) and Calvo (1978), the literature on optimal government policy without commitment formalized interactions between a large player (government) and a fringe of small players (households, lenders), dynamic policy games, by building on the tools developed in the work of Abreu (1988) and Abreu et al. (1990) in the literature of repeated games. This agenda has studied interesting applications for capital taxation (e.g., Chari and Kehoe, 1990, Phelan and Stacchetti, 2001, Farhi et al., 2012), monetary policy (e.g., Chang, 1998, Sleet, 2001, Waki et al. 2018) and sovereign debt (e.g., Eaton and Gersovitz, 1981, Dovis, 2019) and helped us to understand the distortions introduced by lack of commitment and the extent to which governments can rely on reputation to achieve better outcomes.

One of the challenges in studying dynamic policy games is that these settings typically feature a wide range of equilibria with different predictions over observable outcomes. For example, there are "good" equilibria where the government may achieve, or come close to achieving, the optimum with commitment, while there are "bad" equilibria where this is far from the case, and the government may be playing the repeated static best response. When studying dynamic policy games, which of these should we expect to be played? Can we make general predictions given this pervasive equilibrium multiplicity? One approach is imposing refinements, such as various renegotiation-proof notions, that either select an equilibrium or significantly reduce the set of equilibria. Unfortunately, no general consensus has emerged on the appropriate refinements.

Our goal is to overcome the challenge multiplicity raises by providing predictions in dynamic policy games that hold across all equilibria; following the terminology of Bergemann and Morris (2013), *robust predictions*. The approach we offer involves making predictions for future play that depend on past, observed play. The key idea is that even when little can be said about the *unconditional* path of play, quite a bit can be said once we *condition* on past observations. To the best of our knowledge, this simple idea has not been exploited as a way of deriving robust implications from the theory. Formally, we introduce and study a concept which we term "equilibrium consistent outcomes": outcomes of the game, after an observed history, that are consistent with some subgame perfect equilibria (SPE) that on its path could have generated the observed history.

Although the notions we propose and the results we derive are general and apply to a large class of dynamic policy games, for concreteness we first develop them for a specific application, using a model of sovereign debt along the lines of Eaton and Gersovitz (1981). In the model, a small open economy faces a stochastic stream of income. To smooth consumption, a benevolent government can borrow from international debt markets, but lacks commitment to repay. If it defaults on its debt, the only punishment is permanent exclusion from financial markets; it can never borrow again. There are two features of this model that make it appealing to our work. First, this model has been widely adopted and is a workhorse in international economics. Second, this policy game can feature wide equilibrium multiplicity. On one end of the spectrum, in the worst equilibrium, the government is in autarky, facing a price of zero for debt issuance, and consuming its income. Meanwhile, in the best equilibrium, the government smooths consumption, and there is no room for self-fulfilling crises.

Our main result, Proposition 1, following the classic approach to study correlated equilibrium first proposed by Aumann (1987),[1] characterizes probability distributions over outcomes, what we term as *equilibrium consistent distributions*. Even though in the model any *equilibrium* price can be realized after a particular equilibrium history, we show that there are bounds on the probability distributions over these prices. For example, if the country just repaid a high amount of debt, or did so under harsh economic conditions (e.g., when output was low), then low price realizations are less likely. The choice to repay under these conditions reveals an optimistic outlook for bond prices that narrows down the set of possible equilibria for the continuation game. This optimistic outlook is the expression of a *dynamic revealed preference* argument. What the government has left on the table as a consequence of its past decisions, reveals its expectations over future play. In equilibrium, these expectations must be correct, and hence they impose restrictions over expected future outcomes, which form the basis of our predictions.

Building on the characterization of equilibrium consistent distributions, we next explore the predictions on all moments of observables that hold across all equilibria. In particular, we focus on debt prices. First, in Proposition 2, we obtain bounds on the maximum probability of low prices; for example, a rollover debt crises (i.e., a price realization of zero). Due to equilibrium multiplicity, rollover debt crises may occur on the equilibrium path for any realization of the fundamentals. However, the probability of a rollover crisis, after a certain history, may be constrained. We derive these constraints, showing that rollover crises are less likely if the borrower has recently made sacrifices to repay. Second, we use our characterization to obtain bounds on moments of distributions over outcomes. In particular, in Proposition 3, we characterize bounds over the expected value of debt prices given a history for any equilibrium. Third, in Proposition 4, as in Bergemann et al. (2015), we characterize bounds on variances that hold across all equilibria. In

[1] More recently, this approach has been also adopted by the literature on information design. See Bergemann and Morris (2018) for a review.

addition, as a Corollary of these three propositions, we propose a simple linear program that characterizes all non-centered moments over observables. Finally, in Proposition 5, we extend Proposition 1 for the case in which government policies are state contingent. The importance of this case is that it allows us to study the joint behavior of government policies, prices, and the driving forces of the model. For example, we can obtain bounds on the maximal variance subject to a constraint on the co-variance of capital flows and output.[2]

In the last section of the paper, Section 4, we show how our characterization of equilibrium consistent outcomes extends to a more general class of dynamic policy games. In particular, we provide a general model of credible government policies, which follows the seminal contribution of Stokey (1991). The key features that the general setup tries to capture are lack of commitment, a time inconsistency problem, an infinite horizon that creates reputation concerns in the sense of trigger-strategy equilibria, and short run players that form expectations regarding government policies. After laying out the general model, we show how the model of sovereign debt as in Eaton and Gersovitz (1981) and the New Keynesian model as in Galí (2015) fit in this setup and we then discuss how the main results of the paper, Proposition 1, extends into this general environment. In addtion, in Section 4 we also study a variation of the model in which not all defaults are punished with permanent reversion to autarky, in the spirit of Grossman and Huyck (1989) and more recently in Dovis (2019). In particular, we discuss the extent that the predictions of our paper are robust in the case in which not all defaults are punished with permanent autarky.

Example and Main Results. We illustrate our main results in a simple two-period example. Figure 1 depicts a two player game in which the government has the choice of defaulting (choosing x = Default) and receiving a sure payoff of $\underline{u}$, or repaying debt (choosing x =Repay), and hence choosing to play a simultaneous move game G with the investors. If the government chooses x = Repay, a public random variable $\zeta \sim$ Uniform $[0,1]$ (a *sunspot* variable) is observed by both parties before the subgame to be played between the government and the investors. The choices for the government (debt) and the investors (debt prices) in the coordination game are $(b_h, b_l) \in \mathbb{R}^2$ and $(q_h, q_l) \in \mathbb{R}^2$ respectively. The parametric assumptions are that $u_h, u_l, a, b > 0$ and $\underline{u} \in (u_l, u_h)$.[3]

Equilibrium. The subgame following x = "Repay" has two equilibria in pure strategies:

[2] We thank an anonymous referee for suggesting this extension.

[3] The game that we study in this example is slightly different to the one that we study in Section 2. The coordination game in the second step of the game depicted in Figure 1 tries to illustrate the inherent coordination over continuation play at the heart of repeated games, which is also the cause for the typical equilibrium multiplicity present in these games.

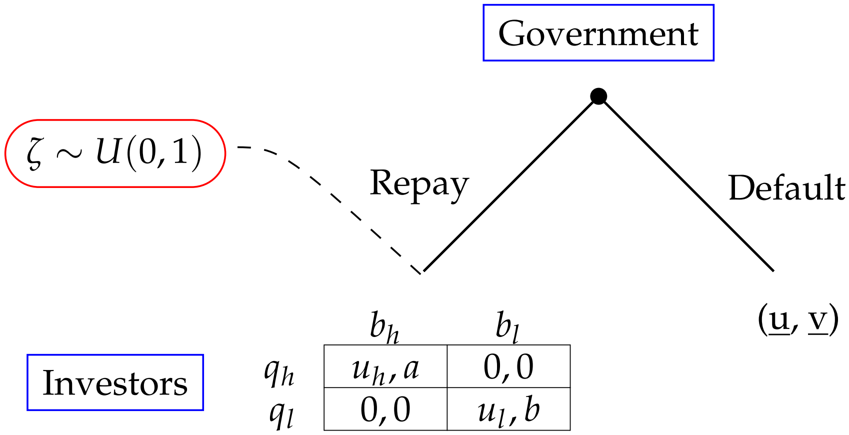

Figure 1: two period Example

(b_l, q_l) and (b_h, q_h), which we will call the low and high equilibria. We can summarize any equilibrium outcome as a pair (x, Q), where $x \in \{\text{Repay}, \text{Default}\}$ is the government's decision whether or not to play the coordination game or not, and $Q = (Q_l, Q_h)$ is a distribution over the low and high equilibrium; i.e., $Q_k = \Pr(\zeta : (b_k, q_k) \text{ is played})$ for $k \in \{l, h\}$, and $Q_l + Q_h = 1$. This class is a subset of the correlated equilibrium distributions of Aumann (1987) for this static subgame.

Equilibrium Consistent Distributions. Our main result, Proposition 1, characterizes distributions over observables after observing a equilibrium history of play. Lets delve into the intuition of this result. Suppose that we (as outsiders) observe that the government has repaid debt. Both the high and low equilibrium are Nash equilibria of the static game. However, not all distributions over the high and low equilibrium could have been generated by a SPE. Thus, *the fact that some subgame perfect equilibria generated the history will place bounds over outcomes*. For example, there is no equilibrium that on its path generates $x =$ Repay and the government and the investors coordinate in the low equilibrium with probability one. The reason is that $x =$ Repay is not optimal for the government if they expect the low equilibrium with probability one.

Following the same logic, we can dig deeper. In particular, the only equilibrium distributions consistent with $x =$ Repay are those that would have made it optimal for the government to plan $x =$ Repay in the first node. Those distributions $Q_l \in [0, 1]$ are characterized by the following condition:

$$u_h(1 - Q_l) + u_l Q_l \geq \underline{u}. \tag{1.1}$$

Equation (1.1) in fact defines the set of all possible distributions over outcomes that are equilibrium consistent with $x = \text{Repay}$. This sequential optimality of choices, is the main insight of Proposition 1, which is the main result of the paper.

Aided by equation (1.1), we can obtain bounds over moments of distributions. Obtaining these bounds is not computationally costly because they solve a *linear program*.

Bounding Moments: Probability of Crisis. What is the maximum probability of the low equilibrium after observing $x = \text{Repay}$? It is equal to the maximum Q_l, such that (1.1) holds. This value is equal to $\underline{Q}_l := (u_h - \underline{u}) / (u_h - u_l) \in (0, 1)$. This bound is intuitive. As the utility of the good equilibrium u_h increases, $\underline{Q}_l$ increases. As the utility of default $\underline{u}$ increases, this probability decreases. We characterize this bound for the general model in Proposition 2.

Expectations. We also can obtain price expectations. We denote by $\mathbb{E}^Q(q)$ the expected value of the price q for any equilibrium consistent outcome $(x = \text{In}, Q)$. The upper bound, the maximum expectation, is the one that corresponds to the largest probability of the high equilibrium. This probability distribution sets Q_l equal to zero, and has an associated expectation equal to q_h. The lowest expectation solves the following program

$$\underline{\mathbb{E}}^Q(q) = \min_{Q_l} Q_l q_l + (1 - Q_l) q_h$$

subject to (1.1). The solution of this program, and the fact that the largest expectation is q_h, defines a set of expected prices equal to $\left[\underline{\mathbb{E}}^Q(q), q_h\right]$, with $\underline{\mathbb{E}}^Q(q) = \left(1 - \underline{Q}_l\right) q_h + \underline{Q}_l q_l > q_l$. We use the same argument in Proposition 3, where we obtain precise bounds over expectations for the model of sovereign borrowing.

Variances. Once we know the set of all possible expected values of q across equilibria, we also can bound second moments. In particular, we can map distributions over prices q to pairs of expectations and variances $(\mathbb{E}(q), \mathbb{V}(q))$, where $\mathbb{V}(q)$ is the variance of q under some equilibrium distribution Q. In particular, given an expected price $\mu = \mathbb{E}(q) \in \left[\underline{q}, q_h\right]$, the maximum possible variance is $\left(1 - Q_l^\mu\right) q_h^2 + Q_l^\mu q_l^2 - \mu^2$, where $Q_l^\mu := (\mu - q_l) / (q_h - q_l)$. Again, this is the solution to a linear program, in which the objective is the variance, and the constraint is, (1.1), and the fact that the mean of the distribution is equal to μ. In Proposition 4, we show that for the model of sovereign borrowing, the upper bound on variance always solves a linear programming problem as well, and actually can always be implemented by a distribution with only two prices in its support (even if q is a continuum).

Equilibrium Consistency vs. Forward Induction. It is important to distinguish equilibrium consistency from Forward Induction. The game depicted in Figure 1 is also useful for that.

For concreteness, suppose that there is no sunspot (i.e. ζ, is constant). In this game, the set of subgame perfect equilibria with forward induction has only one equilibrium in pure strategies ($x = \text{Repay}, (b_h, q_h)$). The subgame perfect equilibrium ($x = \text{Default}, (b_l, q_l)$) does not survive forward induction. But, because it is a subgame perfect equilibrium, it is equilibrium consistent. This example illustrates the main difference between the two solution concepts. Forward induction is a refinement on the set of equilibria; i.e., it shrinks the set of subgame perfect equilibria. Equilibrium consistency, on the other hand, does not shrink the set of equilibria, but rather introduces restrictions on observables.

Literature Review. Our paper relates to several strands of the literature. First, to the literature on credible government policies. The seminal papers on optimal policy without commitment are Kydland and Prescott (1977) and Calvo (1978).[4] We believe that our paper is closely related to Chari and Kehoe (1990), Stokey (1991), and Atkeson (1991). The first two papers adapt the techniques developed in Abreu (1988) to characterize completely the set of equilibria in dynamic policy games. Atkeson (1991) extends the techniques in Abreu et al. (1990) by allowing for a stochastic public state variable, in the context of sovereign lending, finding properties of the best equilibrium. We study a related, yet different question. Instead of characterizing equilibria at the ex-ante stage of the game in terms of sequences of observables, we provide a recursive characterization of the set of continuation equilibria given an equilibrium history of play. This characterization of continuation equilibria is the basis for obtaining predictions that are robust across all equilibria. Our central assumption is that *an* equilibrium generates the history of play, without appealing to any equilibrium refinement.

Second, to the literature on robust predictions. The papers that are more closely related to our work are Angeletos and Pavan (2013), Bergemann and Morris (2013), and Bergemann et al. (2015). The first paper, Angeletos and Pavan (2013), obtains predictions that hold across every equilibrium in a global game with an endogenous information structure. The second paper, Bergemann and Morris (2013), in a class of coordination games with normal public and private signals about a payoff-relevant state variable, obtains restrictions over moments of observable endogenous variables that hold across every possible information structure. In a related paper, Bergemann et al. (2015) characterize bounds on output volatility across all potential information structures in a static model where agents face both idiosyncratic and aggregate shocks to productivity.

[4] Applications range from capital taxation as in Phelan and Stacchetti (2001); monetary policy as in Chang (1998) and Waki et al. (2018); and sovereign debt as in Atkeson (1991), Arellano (2008), Aguiar and Gopinath (2006), and more recently Dovis (2019).

Our paper contributes to this literature by obtaining predictions that hold across all equilibria in a dynamic game. Differently from Bergemann and Morris (2013), in our environment, there is no payoff relevant private information. However, this simplification allows us to focus on a class of dynamic policy games with exogenous and endogenous state variables. In the application we focus on this paper, we obtain restrictions over the distribution of equilibrium debt prices, for any possible process of sunspots (potentially non-stationary), by exploiting the dynamic implications that sequential rationality has on the distribution of observables. These implications are the basis to obtain bounds on first and second order conditional moments, across all possible sunspot processes, or following the terminology in Bergemann and Morris (2018), across all possible *information structures*.

The literature of information design in dynamic games, where agents may have access to private information about other players actions, was first formalized by Myerson (1986) and Forges (1986), extending the concept of correlated equilibrium of Aumann (1987) to extensive form games. As reviewed in Bergemann and Morris (2018), one can view the problem of information design from two alternative points of view. In the first one, the "literal interpretation", an information designer sends signals to other parties, to influence their behavior in order to achieve some objective. A large literature has grown after the contribution of Kamenica and Gentzkow (2011); see for example, on static environments, Gentzkow and Kamenica (2014), among others. In the second one, the "metaphorical interpretation", the designer is an abstraction that chooses among different information structures to achieve some objective. For example, in Bergemann et al. (2015), the "objective" of the designer is to maximize output volatility. The literature on robust predictions falls in this category; see for example Bergemann and Morris (2013). Our paper belongs to the second interpretation.

Chahrour and Ulbricht (2020) use this approach while extending their results to dynamic linear macroeconomic environments, where agents have access to arbitrary dynamic information structures about fundamental shocks and prices. The authors also obtain moment conditions on "wedges" that are akin to the results in Bergemann and Morris (2013) and ours as well, which allows them to obtain testable implications. In our paper, we instead focus on pure strategic uncertainty rather than payoff uncertainty. Also related is De Oliveira and Lamba (2019), where the authors obtain testable implications of Bayesian rationality over a single agent choosing sequentially, but where agents may have access to an arbitrary dynamic information structures that could rationalize their behavior. These bounds provide testable implications of the model, even in the presence of both equilibrium multiplicity and uncertainty of the information structure agents have when making their decisions.

Third, sections 2 and 3 of this paper study robust predictions in a dynamic policy game that builds on Eaton and Gersovitz (1981). This framework, and variations of it, have been extensively used to study sovereign borrowing following the initial contributions of Aguiar and Gopinath (2006) and Arellano (2008). The focus is usually on Markov equilibria on payoff relevant state variables and hence defaults can only be a consequence of bad fundamentals. Our paper shares with this strand of the literature the focus on a model along the lines of Eaton and Gersovitz (1981), but rather than characterizing a particular equilibrium, we study predictions across all equilibria.

Outline. The paper is structured as follows. We introduce the model in Section 2. In Section 3 we discuss the characterization of equilibrium consistent outcomes. In Section 4, we present a general dynamic policy game and state the main results of the paper in this more general setup. We conclude in Section 5.

2 A Dynamic Policy Game

Our model of sovereign debt follows Eaton and Gersovitz (1981). Time is discrete and denoted by $t \in \{0, 1, 2, ...\}$. A *small open economy* receives a stochastic stream of income denoted by y_t. Income follows a Markov process with c.d.f. denoted by $F(y_{t+1} \mid y_t)$, with finite moments. The c.d.f. $F(y_{t+1} \mid y_t)$ is non-atomic. There is a public randomization device, $\zeta_t \sim U[0, 1]$, i.id. over time. The government is benevolent and seeks to maximize the utility of the households. It does so by selling bonds, denoted by b_t, in the international bond market. The household evaluates consumption streams according to:

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right],$$

where $\beta < 1$ and u is increasing, strictly concave and bounded below.[5] The sovereign government issues short term debt at a price q_t. The budget constraint is:

$$c_t = y_t - b_t + q_t b_{t+1}.$$

There is limited enforcement of debt. Therefore, the government will repay only if it is more convenient to do so. We assume that after a default the government remains in

[5]We introduce the assumption that the utility function is bounded to guarantee that the value function is finite.

autarky forever after but there are no direct output costs of default. Furthermore, we also assume that the government cannot save:

$$b_{t+1} \geq 0.$$

There is a competitive fringe of risk neutral *investors* that discount the future at a rate of $r > 0$. This discount rate, and the possibility of default, imply that the price of the bond is given by:

$$q_t = \frac{1-\delta_t}{1+r}, \tag{2.1}$$

where δ_t if the default probability on bonds b_{t+1} issued at date t.[6]

Timing. In period t, the government enters with b_t bonds that it needs to repay. Then income y_t is realized. The government then has the option to default $d_t \in \{0,1\}$. If the government does not default, the government runs an auction of face value b_{t+1}. A sunspot variable ζ_t, which is common knowledge and independent of y_t, is realized. Then, the price of the bond q_t is realized. Finally, consumption takes place, and is given by $c_t = y_t - b_t + q_t b_{t+1}$.[7] If the government decides to default, then consumption is equal to income, $c_t = y_t$. The same is true if the government has ever defaulted in the past.

Histories, Strategies, and Outcomes. A *history* is a vector $h^t = (h_0, h_1, ..., h_{t-1})$, where $h_t = (y_t, d_t, b_{t+1}, \zeta_t, q_t)$ is the outcome of observable variables of the stage game at time t. A partial history is an initial history h^t concatenated with a history of the stage game at period t. For example, $h^t_g = (h^t, y_t)$ is a history after which the government must choose policies (d_t, b_{t+1}). The set of all partial histories is denoted by $\mathcal{H}$. We label as $\mathcal{H}_g \subset \mathcal{H}$ the partial histories where the government has to choose policies. Likewise, $\mathcal{H}_{m,\zeta} \subset \mathcal{H}$ is the set of partial histories where the market plays; i.e., $h^t_{m,\zeta} = (h^t, y_t, d_t, b_{t+1}, \zeta_t)$. We denote the histories where the market plays but the sunspot has not been realized by h^t_m, so $h^t_{m,\zeta} = (h^t_m, \zeta_t)$. A policy maker's strategy is a function $\sigma_g(h^t, y_t) = \left(d^{\sigma_g}_t, b^{\sigma_g}_{t+1}\right)$ for all histories $(h^t, y_t) \in \mathcal{H}_g$. A strategy for the market is a pricing function $q_m(h^t, y_t, d_t, b_{t+1}, \zeta_t)$ for all histories $h^t_{m,\zeta} \in \mathcal{H}_m$. We denote by Σ_g and Σ_m the set of strategies for the government and the market. For a strategy profile $\sigma = (\sigma_g, q_m)$, we write $V(\sigma \mid h)$ for the

[6] This can be micro-founded by a fringe of strategic agents who decide to lend b_{t+1} dollars to maximize expected profits $V = -q_t b_{t+1} + (1-\delta_t)\frac{1}{1+r} b_{t+1}$. If agents compete perfectly in the lending market, equation 2.1 is derived as a non-arbitrage equilibrium condition. See for example Arellano (2008).

[7] If the realized price at the auction is such that the budget constraint does not hold, the government can access funds to guarantee that consumption equals zero (i.e., such that the budget constraint holds ex-post). However, due to accessing these special funds, in this case, utility is equal to $-\infty$.

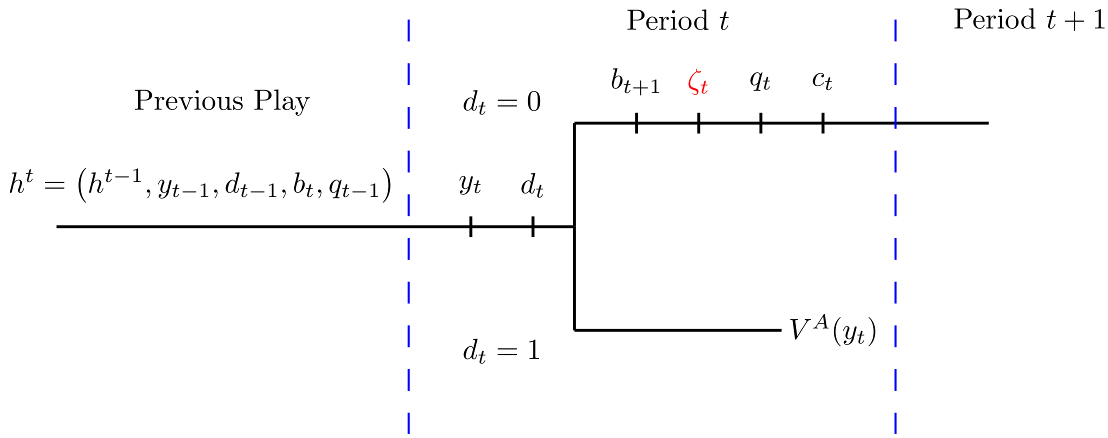

Figure 2: The figure summarizes the timing and the construction of histories in the case in which there is a sunspot. Now, we introduce a sunspot ζ_t after the government has issued debt b_{t+1} and before the price q_t has been realized.

continuation expected utility, after history h, of the representative consumer if agents play according to profile σ. For any strategy profile $\sigma \in \Sigma := \Sigma_g \times \Sigma_m$, we define the continuation at $h_g^t \in \mathcal{H}_g$:

$$V(\sigma \mid h_g^t) = \mathbb{E}_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[(1 - d_s^t) u(y_s - b_s + q_s b_{s+1}) + d_s u(y_s) \right] \right\},$$

where (y_s, d_s, b_{s+1}, q_s) are generated by the strategy profile σ.[8]

Equilibrium. A strategy profile $\sigma = (\sigma_g, q_m)$ constitutes a *subgame perfect equilibrium (SPE)* if and only if, for all partial histories, $h_g^t \in \mathcal{H}_g$:

$$V(\sigma \mid h_g^t) \geq V(\sigma_g', q_m \mid h_g^t) \text{ for all } \sigma_g' \in \Sigma_g, \tag{2.2}$$

and for all histories $h_{m,\zeta}^t$, it holds that:

$$q_m \left(h_{m,\zeta}^t \right) = \frac{1}{1+r} \mathbb{E}_t \left(1 - d^{\sigma_g}(h^{t+1}, y_{t+1}) \right). \tag{2.3}$$

[8]Note that expectation is taken with respect to the probability distribution of the stochastic processes of output and the sunspot, given the strategy for both the market and the government. We sometimes use $b_s = b_s^{\sigma_g}$ and $d_s = d_s^{\sigma_g}$ for clarity.

That is, the strategy of the government is optimal given the pricing strategy of the lenders $q_m(\cdot)$; likewise, $q_m(\cdot)$ is consistent with the default policy generated by σ_g. The set of all subgame perfect equilibria is denoted as $\Sigma^* \subset \Sigma$. Given any history $h \in \mathcal{H}$, we denote $\Sigma^*(y_t, b_{t+1})$ as the set of all equilibrium strategies of the subgame starting at h^t.[9]

Equilibrium Prices, Continuation Values. For any history h_m^t, we define the highest and lowest *equilibrium* prices as:

$$\overline{q}^E(h_m^t) := \max_{q_m \in \Sigma^*(h_m^t)} q_m\left(h_{m,\zeta}^t\right) \tag{2.4}$$

$$\underline{q}^E(h_m^t) := \min_{q_m \in \Sigma^*(h_m^t)} q_m\left(h_{m,\zeta}^t\right). \tag{2.5}$$

The worst SPE price is zero (i.e., $\underline{q}^E(h_m^t) = 0$) and the associated equilibrium payoff is given by the utility level of autarky. The lowest price $\underline{q}^E(h_m^t)$ is attained by using a fixed strategy for all histories (default after any history). The level of utility of autarky is given by:

$$V^A(y_t) := u(y_t) + \beta \mathbb{E}_{y_{t+1}|y_t} V^A(y_{t+1}). \tag{2.6}$$

Alternatively, the highest price $\overline{q}^E(h_m^t)$ is associated with a, different, fixed strategy for all histories, is Markov in (b_t, y_t) conditional on no default so far, and delivers the highest equilibrium level of utility for the government.[10] We denote the best equilibrium price as $\overline{q}^E(h_m^t) = \overline{q}(y_t, b_{t+1})$. The continuation utility (conditional on not defaulting) of the choice b_{t+1} given bonds and output (b_t, y_t) in the best equilibrium is given by:

$$V^{nd}(b_t, y_t, b_{t+1}) := u\left(y_t - b_t + \overline{q}(y_t, b_{t+1})\, b_{t+1}\right) + \beta \overline{\mathbb{V}}(y_t, b_{t+1}), \tag{2.7}$$

where $\overline{\mathbb{V}}(y_t, b_{t+1})$ is defined as

$$\overline{\mathbb{V}}(y_t, b_{t+1}) := \mathbb{E}_{y_{t+1}|y_t}\left[\max\left\{\overline{V}^{nd}(b_{t+1}, y_{t+1}), V^A(y_{t+1})\right\}\right], \tag{2.8}$$

and $\overline{V}^{nd}(b_t, y_t) := \max_{b_{t+1} \geq 0} V^{nd}(b_t, y_t, b_{t+1})$. Aided with the previous definitions, the best equilibrium price is defined as $\overline{q}(y_t, b_{t+1}) := \frac{\mathbb{E}_{y_{t+1}|y_t}[1 - d(y_{t+1}, b_{t+1})]}{1+r}$ where $d(y_{t+1}, b_{t+1})$

[9] Note that, as is standard in dynamic games, the history preceding (y_t, b_{t+1}) does not restrict the set of equilibria after that history.

[10] In the Online Appendix of Passadore and Xandri (2020) we describe necessary and sufficient conditions for equilibrium multiplicity, and we show that the best SPE is characterized by (2.7) and (2.8) below. See also Auclert and Rognlie (2016), Proposition 6, and Bloise et al. (2017), Section 6, for conditions under which there is equilibrium multiplicity.

is equal to zero if and only if $\overline{V}^{nd}(b_{t+1}, y_{t+1})$ is greater than of equal to $V^A(y_{t+1})$. This equilibrium that we just described is the one analyzed in the standard Eaton and Gersovitz (1981) model.

Summing up. After the describing the environment, and the best and the worst SPE, in the next Section we prove the main result of the paper: we characterize probability distributions on prices that can be a continuation equilibrium after an equilibrium history. Note that any price in $[0, \bar{q}]$ can be realized (i.e.; is a SPE outcome) after the realization of the sunspot. However, as we will show in Proposition 1, equilibrium histories of play h^t, will place restrictions on distributions of prices. For example, in one of our applications, Proposition 2, we characterize the maximum probability of obtaining low prices, by exploiting the restrictions on distributions of prices that we obtain in Proposition 1. In particular, we obtain formulas to compute:

$$\max_{Q \in \mathbb{ECD}(h^t)} \Pr_Q (q \leq \hat{q}), \tag{2.9}$$

which is the maximum probability that debt prices are lower than $\hat{q}$, after observing the equilibrium history h^t. Characterizing the set $\mathbb{ECD}(h^t)$, which denotes the set of probability distributions that are consistent with an equilibrium history, is the main task for the next section.

3 Equilibrium Consistency

We now introduce the concept of equilibrium consistency. Given a SPE profile $\sigma = (\sigma_g, q_m)$, we define its *equilibrium path* $x(\sigma)$ as a sequence of measurable functions $x(\sigma) = \left(d_t^{\sigma_g}(\zeta^{t-1}, y^t), b_{t+1}^{\sigma_g}(\zeta^{t-1}, y^t), q_t^{q_m}(y^t, \zeta^t)\right)_{t \in \mathbb{N}}$ that are generated by following the profile σ.

Definition 1. A history $h \in \mathcal{H}$ is *equilibrium consistent* if and only if it is on the support of some equilibrium path $x(\sigma)$, for some SPE profile σ.

3.1 Preliminaries

Before delving into the main result of the paper, we will define and characterize the best equilibrium payoff after a history h_m^t, which is a key input for Proposition 1. The maximum continuation value function $\overline{v}(y_t, b_{t+1}, q_t)$ given an income realization y_t, bonds

b_{t+1}, issued at an equilibrium price q_t, is given by:

$$\overline{v}\left(y_t, b_{t+1}, q_t\right) := \max_{\sigma \in \mathbf{\Sigma}^*\left(y_t, b_{t+1}\right)} V\left(\sigma \mid q_t\right).$$

Two remarks. First, note that because $\sigma \in \mathbf{\Sigma}^*\left(y_t, b_{t+1}\right)$, strategies for the government and the market are equilibrium strategies. In particular, prices are consistent with default policies for every history. For the case of d_{t+1} and q_t, the default policies are consistent with the realized price q_t. In appendix B we characterize this payoff, using the standard approach of Abreu et al. (1990). Second, for this definition we are using the fact that $\overline{v}\left(h_m^t\right) = \overline{v}\left(y_t, b_{t+1}, q_t\right)$. That is, the best continuation payoff after history h_m^t only depends on $\left(y_t, b_{t+1}, q_t\right)$. The next Lemma, provides the characterization and properties of $\overline{v}$, which will be useful to prove the main results in the paper. For Proposition 1, the following Lemma will be useful:

Lemma 1. *(a)* $\overline{v}\left(y_t, b_{t+1}, q_t\right)$ is non-increasing in b_{t+1}, and non-decreasing and concave in q_t. (b) It can be computed as:

$$\overline{v}\left(y_t, b_{t+1}, q_t\right) = \max_{d(\cdot)\in\{0,1\}^Y} \mathbb{E}_{y_{t+1}|y_t}\left[d\left(y_{t+1}\right) V^A\left(y_{t+1}\right) + \left(1 - d\left(y_{t+1}\right)\right) \overline{V}^{nd}\left(b_{t+1}, y_{t+1}\right)\right] \tag{3.1}$$

subject to

$$q_t = \frac{\mathbb{E}_{y_{t+1}|y_t}\left(1 - d\left(y_{t+1}\right)\right)}{1+r}.$$

Proof. See appendix section B. □

The fact that the function is non-increasing in b_{t+1} follows from the fact that the payoff $\overline{V}^{nd}\left(b_{t+1}, y_{t+1}\right)$ is non-increasing in b_{t+1}, which is standard result in the literature that follows Eaton and Gersovitz (1981). The fact that the function is non-decreasing in q_t follows from two facts. First, higher prices are associated with better continuation equilibrium in which the government default in less states of nature. Second, because $b_{t+1} \geq 0$, contemporaneous consumption is higher when q is higher. Finally, concavity follows from the fact that $\overline{v}\left(y_t, b_{t+1}, q_t\right)$ solves a linear programming problem.[11] We use these three properties of $\overline{v}\left(y_t, b_{t+1}, q_t\right)$ to characterize the set of equilibrium consistent distributions and to obtain testable predictions.[12]

[11] Note that because it is a linear program (linear objective and linear constraints), if there is an optimum, it is in the boundaries. Thus, we can solve a relaxed version of the problem, in which $d_t \in [0,1]$, instead of $d_t \in \{0,1\}$. This relaxed problem has a convex feasible set. Thus, for $q_- = q_\lambda := \lambda q_0 + (1-\lambda) q_1$ it holds that $\overline{v}\left(y_-, b, q_\lambda\right) \geq G\left[d_\lambda\left(\cdot\right)\right] = \lambda \overline{v}\left(y_-, b, q_0\right) + (1-\lambda) \overline{v}\left(y_-, b, q_1\right)$, where the inequality comes from the fact that the combination of the optimal policies $d_\lambda =: \lambda d_0 + (1-\lambda) d_1$ is feasible at q_λ.

[12] We will use interchangeably the notation $\overline{v}\left(y_{t-1}, b_t, q_{t-1}\right)$ or $\overline{v}\left(y_t, b_{t+1}, q_t\right)$, depending on what is more

Discussion. In the model that we discussed in Section 2, all defaults imply reversion to permanent autarky, which is the worst equilibrium of the game. We do so to stay close to the literature on sovereign debt which builds on Eaton and Gersovitz (1981). However, it does not need to be the case that all defaults are followed by permanent autarky. For this reason, in Section 4 we study a variation of the model in which debt is state contingent. In this variation the worst subgame perfect equilibrium will still be autarky, but the best continuation equilibrium value, $\overline{v}(y_t, b_{t+1}, q_t)$, which we just characterized in equation (3.1), will be different. As a result of this different best continuation equilibrium value, as we will see in Proposition 1, the predictions across all equilibria will *be quantitatively* altered by alternative assumptions of what happens after a default.

3.2 A Characterization

The main objective of the paper in characterizing equilibrium consistent policies and distributions over prices in t. Formally, a distribution of debt prices $Q_t \in \Delta(\mathbb{R}_+)$ is equilibrium consistent with history h_m^t if and only if for any Borel measurable set of prices $A \subseteq \mathbb{R}_+$ we have that $Q_t(A) = \Pr\left(\zeta_t : q_m\left(h_{m,\zeta}^t\right) \in A\right)$ for some $q_m \in \Sigma_m^*(y_t, b_{t+1})$. Denote the set of *equilibrium consistent price distributions* as $\mathbb{ECD}\left(h_m^t\right)$. A triple $(d_t = 0, b_{t+1}, Q_t)$ is an *equilibrium consistent outcome* if and only if there exists an equilibrium profile $\sigma = (\sigma_g, q_m)$ that generates on its path $(d_t = 0, b_{t+1})$ and the distribution of prices Q_t.[13] Armed with these definitions we will now characterize the implications of equilibrium consistency on observables.

Proposition 1. *Suppose that $\left(h^t, y_t\right)$, with no default so far, is equilibrium consistent. Then, the triple $(d_t = 0, b_{t+1}, Q_t)$, where $Q_t \in \Delta(\mathbb{R}_+)$, is an equilibrium consistent outcome if and only if:*

(a) Debt prices are SPE prices; i.e.,

$$Q_t \in \Delta\left(\left[0, \overline{q}\left(y_t, b_{t+1}\right)\right]\right). \tag{3.2}$$

(b) Incentive compatibility (IC) of the government:

$$\int_0^{\overline{q}(y_t, b_{t+1})} \left[u\left(y_t - b_t + q_t b_{t+1}\right) + \beta \overline{v}\left(y_t, b_{t+1}, q_t\right)\right] dQ_t\left(q_t\right) \geq V^A\left(y_t\right). \tag{3.3}$$

convenient. Note that the set of equilibrium strategies given history h^t, which we denote by $\Sigma^*(h^t)$, only depends on the initial bonds, b_t, and the seed value of income, y_{t-1}. Thus, $\Sigma^*(h^t) = \mathbf{\Sigma}^*\left(y_{t-1}, b_t\right)$. Therefore if $\sigma \in \Sigma^*(y_t, b_{t+1})$ conditional on q_t, must satisfy the property that the goverment's default choices are consistent with the realized price q_t.

[13] Following our focus on observable variables, the corresponding object to a pricing strategy $q_m(h_{m,\zeta}^t)$ is a distribution Q_t, which is why we treat it as an observable physical outcome.

Proof. See Appendix A.1. □

The main contribution of our paper is using condition (3.3) to derive restrictions on equilibrium objects that are consistent across SPE of the policy game. Note that condition (3.2) characterizes prices that are SPE outcomes. Debt prices are between zero, and the best equilibrium price $\overline{q}\left(y_t, b_{t+1}\right)$. The idea is that, if we do not assume that the history h_m^t is generated by some SPE, then there are no restrictions over debt prices other than being equilibrium prices.

The idea of the proof of Proposition 1 is as follows. For necessity, fix an equilibrium consistent distribution Q after history h_m^t. If we assume that h_m^t is on the equilibrium path of some SPE, then the government strategies, d_t and b_{t+1}, were optimal before the realization of the sunspot ζ_t. This implies that the government ex-ante preferred to pay the debt (i.e., $d_t = 0$) and issue bonds (b_{t+1}) rather than defaulting on the debt. If, after these decisions the realized price is q_t, the payoff for the government would be *at mos*t $u\left(y_t - b_t + q_t b_{t+1}\right)$ plus the best ex-post continuation value $\overline{v}\left(y_t, b_{t+1}, q_t\right)$. However, the government is uncertain over which price will be realized for the debt issued. So, the government forms an expectation with respect to the "candidate" equilibrium consistent distribution Q. This expectation, and its associated expected utility, has to be at least as good as defaulting; if not, the government would have defaulted instead of repaying. The left hand side of condition (3.3) is an upper bound on the utility of not defaulting at history h_m^t. Thus, (3.3) is necessary. In other words, if it was violated, then we could not construct promises that rationalize the past history h_m^t.[14]

The idea of sufficiency, which is the reason why we eliminate b_{t-1} and all the previous policies, stems from the fact that both the output and the sunspot are non-atomic.[15] The particular history that followed h_m^{t-1} when b_{t-1} was chosen, the one with the particular realization of ζ_{t-1}, had zero probability of occurring, because the sunspot has a continuous distribution. Thus, it could always have been the case that the payoffs that rationalized b_{t-1} and the previous policies were to be realized in a state that never materialized. Therefore, $\mathbb{ECD}\left(b_t, y_t, b_{t+1}\right) = \mathbb{ECD}\left(h_m^t\right)$.

There are two points that are worth noting regarding alternative assumptions of the game and how robust the predictions are. First, in the model that we developed in Section 2, by assumption, all defaults imply the reversion to the worst equilibrium, which pins down the function $\overline{v}\left(y_t, b_{t+1}, q_t\right)$, characterized in (3.1). In the case in which perma-

[14]One might wonder why we cannot rely on the best continuation payoff $\overline{\mathbb{V}}(y_t, b_{t+1})$. This is because this payoff is associated with the best equilibrium price, and this price needs not to be realized. The best possible payoff, after the price q is realized, is $\overline{v}\left(y_t, b_{t+1}, q_t\right)$.

[15]Even if output was discrete, sunspots make shocks non-atomic, having the same effect as if we had absolutely continuous output shocks.

nent autarky is the worst equilibrium, alternative assumptions of what happens after a default, will imply different characterization of the best continuation equilibrium (3.1), and will affect predictions via (3.3). To clarify this case, in Section 4, we study the case with excusable defaults as in Grossman and Huyck 1989, where on the equilibrium path defaults do not trigger punishments.

Second, Proposition 1 can be specialized to obtain robust predictions over a certain subset of subgame perfect equilibrium. For example, the result can be adapted for equilibria with limited equilibrium punishments. Namely, the same results would hold if we replace the worst equilibrium of the game $V^A(y)$ by a higher equilibrium payoff $V > V^A(y)$ in the characterization of the best continuation equilibrium in equation (3.1) and on the right hand side of (3.3). An example is the case in which agents are punished after every default with a fixed (or random number of periods) in autarky.

3.3 Bounding Certain Prices

Aided by Proposition 1, we can now further characterize moments over distributions of debt prices. Before bounding moments over distributions of prices we characterize the best continuation prices, for the case without sunspots; i.e., ζ_t is constant. We term them *certain equilibrium consistent* prices. First, for each (b_t, y_t, b_{t+1}), we define the lowest (certain) equilibrium consistent price, $\underline{q}(b_t, y_t, b_{t+1})$, as the solution $\underline{q}$ to:

$$u\left(y_t - b_t + \underline{q}b_{t+1}\right) + \beta\overline{v}\left(y_t, b_{t+1}, \underline{q}\right) = V^A(y_t). \tag{3.4}$$

Note that $\underline{q}(\cdot)$ is a function that maps $\underline{q}(b_t, y_t, b_{t+1}) : B \times Y \times B \to \left[0, \frac{1}{1+r}\right]$. Note also that $\underline{q}$ is unique, due to the monotonicity of $u(\cdot)$ and $\overline{v}(y_t, b_{t+1}, \cdot)$. The lowest (certain) equilibrium consistent price, $\underline{q}$, is the lowest price for debt issued b_{t+1}, given a debt payment b_t under an income realization y_t, for which the government does not default. Second, we can also define the highest equilibrium consistent price. It is given by $\overline{q}(y_t, b_{t+1})$, and is equal to the best equilibrium price defined in equation (2.4). The idea is that for any equilibrium history, the best equilibrium is a possible continuation equilibrium. In fact, if the best equilibrium is not a possible continuation, then the previous history cannot be an equilibrium history. Next, we show some properties about these prices.

Corollary 1. *Let $\underline{q}(b_t, y_t, b_{t+1})$ be the lowest (certain) equilibrium consistent price after history h^t_m. The following holds: (a) $\underline{q}(b_t, y_t, b_{t+1})$ is increasing in b_t; (b) for every equilibrium consistent history, $-b_t + \underline{q}(b_t, y_t, b_{t+1})\, b_{t+1} \leq 0$; (c) if income is i.i.d., then $\underline{q}$ is decreasing in y_t, and so is the set of (certain) equilibrium consistent prices $[\underline{q}(b_t, y_t, b_{t+1}), \overline{q}(y_t, b_{t+1})]$.*

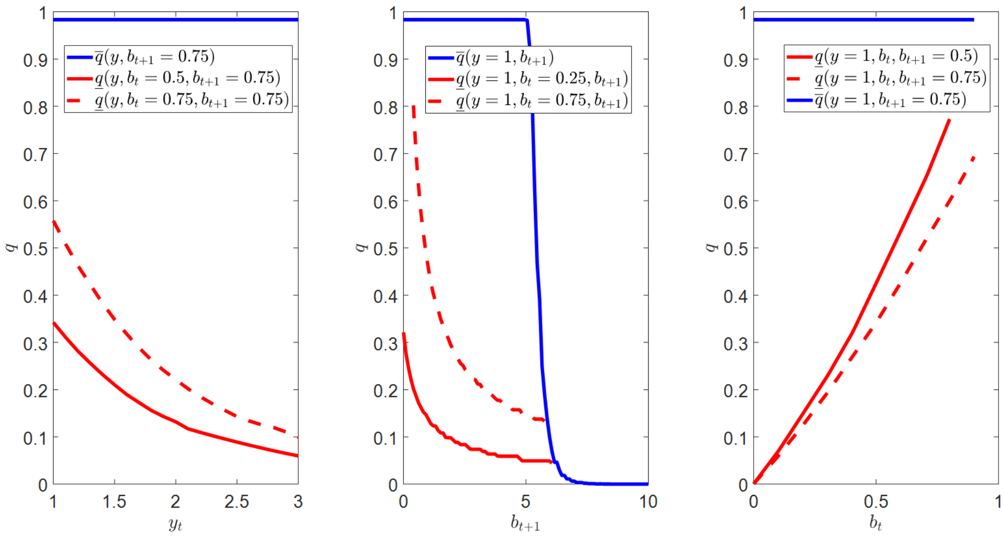

Figure 3: This figure shows (certain) equilibrium consistent prices $\overline{q}$ and $\underline{q}$. We describe the comparative statistics after history h_m^t. Thus, the relevant state variables are (b_t, y_t, b_{t+1}).

Proof. See Appendix A.2. □

The intuition for Corollary 1 follows. First, note that if the government just repaid a large amount of debt (i.e., made an effort to repay the debt), then the past choices are rationalized by higher continuation prices, which is a result of the fact that the utility function is increasing in consumption and that the best continuation is increasing in prices. Second, note that a positive capital inflow obtained at the lowest (certain) equilibrium consistent prices would imply that $u(y_t) - u\left(y_t - b_t + \underline{q}(b_t, y_t, b_{t+1})\, b_{t+1}\right)$ is negative. Intuitively, the country is not making any effort to repay the debt. Therefore, it need not be the case that the country expects high prices for debt in the next period. Finally, because there are no capital inflows at the lowest (certain) equilibrium consistent prices, repaying debt at this price will become more costly for a lower realization of income y_t; this due to the concavity of the utility function. Mathematically, because of concavity, $u(y_t) - u\left(y_t - b_t + \underline{q}(b_t, y_t, b_{t+1})\, b_{t+1}\right)$ is increasing as income decreases, and therefore, the promise-keeping constraint tightens as income decreases.

A Quantitative Illustration. We now numerically solve for the (certain) equilibrium consistent prices. The process for log output is given by $\log y_t = \mu + \rho_y \log y_{t-1} + \sigma_y \epsilon_t$ where $\mu = 0.75$, $\sigma_y = 0.3025$, ϵ_t is i.i.d. and $\epsilon_t \sim N(0,1)$, and $\rho_y = 0.0945$. The risk-free interest rate is set to $r = 0.017$. The utility function is $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, the coefficient of relative risk aversion is $\gamma = 2$, and the discount factor $\beta = 0.953$. Figure 3 depicts the numerical results. As we discussed before, the best equilibrium, $\overline{q}$, coincides with the equilibrium usually studied in the quantitative literature of sovereign debt. We plot the best equilibrium consistent price in blue and the lowest in red. As shown in Figure 3, for low levels of debt the best equilibrium is risk-free (default). As we increase the level of debt, the price drops, and prices drop sharply, as it is in most models with short-term debt (prices are volatile). The lowest (certain) equilibrium consistent prices $\underline{q}(b_t, y_t, b_{t+1})$ are computed using equations (3.1) and (3.4). Note that the comparative statics that we specify in Corollary 1 clearly emerge in Figure 3. First, in the left panel, when the government repays debt $b_t = 0.5$ and issues $b_{t+1} = 0.75$, the lowest (certain) equilibrium consistent prices decrease with the realization of income. In addition, as one would expect, when the amount of debt repaid climbs to $b_t = 0.75$ and the amount of debt issued is still $b_{t+1} = 0.75$, the red dotted line dominates the red line. The lowest (certain) equilibrium consistent prices are now higher. Finally, note that the best equilibrium price is constant through the realizations of income, because for those levels of debt, $b_{t+1} = 0.75$, default is not a concern. Also, note that in the right panel, we observe that with debt repayment, b_t, we obtain the opposite: when the government repays a larger amount of debt, then the lowest (certain) equilibrium consistent prices increases. This is the case for both $(y_t = 1, b_{t+1} = 0.50)$ and $(y_t = 1, b_{t+1} = 0.75)$. The dotted line corresponds to a higher debt issuance, and as we just discussed, given a larger capital inflow, the prices are expected to be lower.

3.4 Bounding Price Distributions

We now delve into the implications of Proposition 1 on distributions of debt prices. The first set of implications are over the probability of low prices. In particular, we characterize the maximum probability that a crisis will occur. Second, we provide bounds across all equilibria for the expectation of prices. Third, we also provide bounds across all equilibria for the variance of distributions over prices.[16] Finally, we study the comparative statistics for the set of equilibrium consistent distributions, $\mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$.

Maximum Probability of Crises. We would like to infer the maximum probability (across equilibria) that the government *could* assign to a price lower or equal to $\hat{q}$ in any equilibrium after an equilibrium history h_m^t. Formally, we define the function $\underline{Q}\left(\hat{q}\right)$ as:

$$\underline{Q}\left(\hat{q}; b_t, y_t, b_{t+1}\right) := \max_{Q \in \mathbb{ECD}(b_t, y_t, b_{t+1})} \Pr_Q\left(q \leq \hat{q}\right) \tag{3.5}$$

[16] All of these bounds are independent of the nature of the sunspots (i.e. the distribution of sunspots, its dimensionality, and so on), in the same way as the set of correlated equilibria does not depend on the actual correlating devices.

where $\Pr_Q(q \leq \hat{q}) := \int_0^{\hat{q}} dQ(q)$. Proposition 2 characterizes $\underline{Q}(\cdot)$.

Proposition 2. *Consider an equilibrium consistent history $h_m^t = (h^t, y_t, d_t = 0, b_{t+1})$. (a) For any $\hat{q} \geq \underline{q}(b_t, y_t, b_{t+1})$, $\underline{Q}(\hat{q}; b_t, y_t, b_{t+1}) = 1$. (b) For any $\hat{q} < \underline{q}(b_t, y_t, b_{t+1})$, it holds that:*

$$\underline{Q}(\hat{q}; b_t, y_t, b_{t+1}) = \frac{V^{nd}(b_t, y_t, b_{t+1}) - V^A(y_t)}{V^{nd}(b_t, y_t, b_{t+1}) - [u(y_t - b_t + \hat{q}b_{t+1}) + \beta\overline{v}(y_t, b_{t+1}, \hat{q})]}. \tag{3.6}$$

Proof. See Appendix A.3. □

The idea of the proof is as follows. Lets us start with the case $\hat{q} \geq \underline{q}(b_t, y_t, b_{t+1})$. The reason why $\underline{Q}(\hat{q}; b_t, y_t, b_{t+1})$ is equal to one is intuitive. A probability distribution that places a probability equal to one on $\underline{q}(b_t, y_t, b_{t+1})$ is an equilibrium consistent distribution. For this distribution, $\Pr_Q(q \leq \hat{q})$ is going to be equal to one. Thus, the max $\Pr_Q(q \leq \hat{q})$ over the set of equilibrium consistent distributions is equal to one. The case in which $\hat{q} < \underline{q}(b_t, y_t, b_{t+1})$ is not that simple. Proposition 2 finds the maximum ex-ante probability (before ζ_t is realized) of observing a price q_t, lower than $\hat{q}$, and it is less than one. To relax the IC constraint for the government, condition (3.3), as much as possible, we consider distributions with binary support over $\{\hat{q}, \overline{q}\}$. For these distributions, when $\overline{q}$ is realized, we assign the best continuation equilibria for the government, and when $\hat{q}$ is realized, we assign the best continuation equilibrium after $q = \hat{q}$, which is given by $\overline{v}(y_t, b_{t+1}, \hat{q})$. The expected value for the government under this distribution, which we label $\underline{Q}(\hat{q}; \cdot)$, needs to be as good as defaulting. When we equalize the value of issuing debt with the distribution $\underline{Q}(\hat{q}; \cdot)$ to the value of defaulting, we obtain an equation for $\underline{Q}(\hat{q}; b_t, y_t, b_{t+1})$, which is precisely given by (3.6).

Note that if the income realization is such that $\overline{V}^{nd}(b_t, y_t) = V^A(y_t)$ (i.e., under the best continuation equilibrium, the government is indifferent between defaulting or not, and still does not default), then $\underline{Q}(\hat{q}; b_t, y_t, b_{t+1}) = 0$ for any $\hat{q} < \underline{q}(b_t, y_t, b_{t+1}) = \overline{q}(y_t, b_{t+1})$. The idea is that for these income levels, only $q = \overline{q}(y_t, b_{t+1})$ is an equilibrium consistent price, and the only distribution that is equilibrium consistent places probability one on that price. Note also that $\underline{Q}$ is a cumulative distribution function for q: it is a non-increasing, right-continuous function with a range of $[0, 1]$; hence it implicitly defines a probability measure for debt prices.

Figure 4 presents the function for the maximum probability of low prices, $\underline{Q}(\hat{q}; b_t, y_t, b_{t+1})$, for different states (b_t, y_t, b_{t+1}). In the left panel, the two distributions differ on the income realization under which the government repaid its debt. Lets start with the blue line: the government repaid debt under an income realization (y_t) of 1.36, repaid 0.5 units of debt (b_t), and issued 0.5 units (b_{t+1}). $\underline{Q}(0)$ is approximately 0.7; in other words,

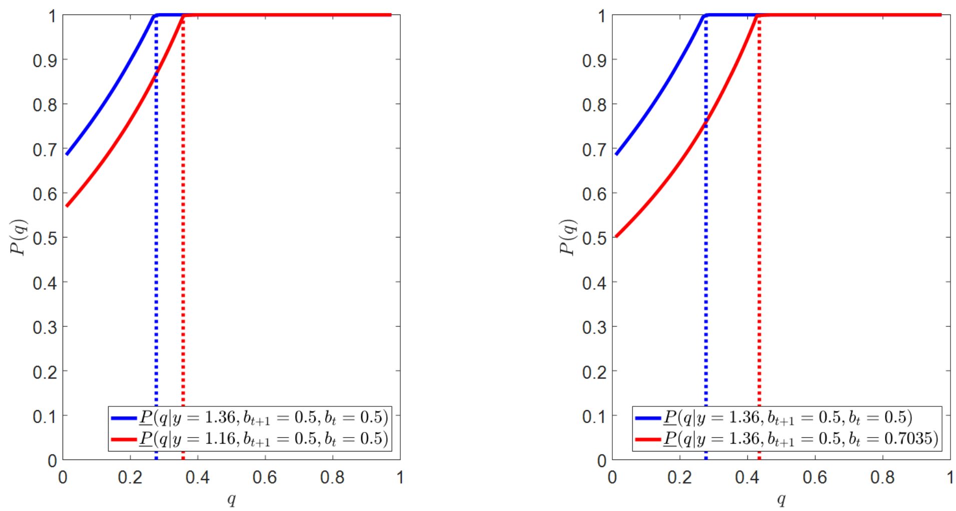

Figure 4: This figure plots $\underline{Q}\,(q)$ for different levels of output for our main calibrated parameters. The left panel fixes $b_{t+1}$and b_t and shows the comparative statistics with respect to y_t. The right panel fixes y_t and shows the comparative statistics with respect to b_t.

the maximum probability of obtaining a price of zero is approximately 0.7. Any distribution where the probability of a price of zero is higher than 0.7, after the history $(b_t, y_t, b_{t+1}) = (0.50, 1.36, 0.50)$, is not equilibrium consistent because it violates the IC constraint of the government. Second, note that as the price $\hat{q}$ increases, $\underline{Q}\,(\hat{q}; b_t, y_t, b_{t+1})$ also increases: the government is willing to accept a higher probability of obtaining low prices (lower than $\hat{q}$), because these prices are not that low. Third, as we should expect, given our previous discussion, the function $\underline{Q}\,(\hat{q}; b_t, y_t, b_{t+1})$ reaches 1 at a price equal to $\underline{q}(b_t, y_t, b_{t+1})\ |_{(b_t,y_t,b_{t+1})=(0.50,1.36,0.50)}$. Fourth, note that the function $\underline{Q}\,(\hat{q})$ shifts if the government repays its debt under poor economic conditions (these conditions imply a lower spot utility); for example, $\underline{Q}\,(0)$ is approximately 0.55 instead of 0.7, if income is 1.16 instead of 1.36, which is what one would expect in order not to violate the incentive compatibility constraint, condition (3.3). Finally, the right hand side of the panel shows the comparative statistics with respect to how much debt is repaid.

Bounding Expectations. One application that is of particular interest is bounding the moments of distributions across all equilibria. We start with expected values. Let $E\,(b_t, y_t, b_{t+1})$ be the the set of all possible $\int q dQ$ for $Q \in \mathbb{ECD}\,(b_t, y_t, b_{t+1})$. The following proposition shows that $E\,(b_t, y_t, b_{t+1})$ is identical to the set of (certain) equilibrium consistent prices when there are no sunspots.

Proposition 3. *Suppose that history $h_m^t = \left(h^t, y_t, d_t, b_{t+1}\right)$ is equilibrium consistent. Then the set of expected prices is equal to the set of certain equilibrium consistent prices (without sunspots); i.e.,*

$$E\left(b_t, y_t, b_{t+1}\right) = \left[\underline{q}\left(b_t, y_t, b_{t+1}\right), \overline{q}\left(y_t, b_{t+1}\right)\right].$$

Moreover, if $b_{t+1} > 0$, then the minimum expected value is uniquely achieved at the degenerate distribution $\hat{Q}$ which assigns probability one to $q = \underline{q}\left(b_t, y_t, b_{t+1}\right)$.

Proof. See Appendix A.4. □

The argument for the proof is based on two facts. First, the monotonicity and the concavity, in q, of the best ex-post continuation value function, $\overline{v}\left(y_t, b_{t+1}, q\right)$. Second, that $\underline{q}\left(\cdot\right)$ is the minimum price, q, for which $u\left(y_t - b_t + qb_{t+1}\right) + \beta\overline{v}\left(y_t, b_{t+1}, q\right)$ is equal to $V^A\left(y_t\right)$.[17] From the second fact, note that the integrand in the left hand side of condition (3.3) is larger than $V^A\left(y_t\right)$ only when q is greater than or equal to $\underline{q}\left(b_t, y_t, b_{t+1}\right)$. The concavity of $\overline{v}\left(y_t, b_{t+1}, q\right)$ and Jensen's inequality then imply that for any distribution $Q \in \mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$, $u\left(y_t - b_t + \mathbb{E}_Q\left(q\right) b_{t+1}\right) + \beta\overline{v}\left(y_t, b_{t+1}, \mathbb{E}_Q\left(q\right)\right)$ has to be greater than or equal to $\int\left[u\left(y_t - b_t + qb_{t+1}\right) + \beta\overline{v}\left(y_t, b_{t+1}, q\right)\right] dQ\left(q\right)$. Because Q is an equilibrium consistent distribution, condition (3.3) implies that the latter needs to be greater than or equal to $V^A\left(y_t\right)$. Thus, because of the monotonicity of $\overline{v}\left(y_t, b_{t+1}, q\right)$, we conclude that $\mathbb{E}_Q\left(q_t\right)$ is greater than (or equal) to $\underline{q}\left(b_t, y_t, b_{t+1}\right)$. The fact that $\mathbb{E}_Q\left(q_t\right)$ is less than or equal to $\overline{q}\left(y_t, b_{t+1}\right)$ is immediate.

Bounding Variances. Next, we characterize bounds over variances. The importance of this application comes not only from the fact that we can obtain dynamic implications from equilibria; we can also know, ex-ante, how much volatility the model can generate. Note that without any a-priori knowledge this can be a daunting task. Which equilibrium will yield the highest variance? In the next proposition, we can pin down how much variance the model can generate, without trying every possible equilibrium. Take any $Q \in \mathbb{ECD}(h_m^t)$ with $\mathbb{E}_Q\left(q_t\right) = \mu$. Denote by $S\left(h_m^t, \mu\right)$ the set of variances of these distributions.

Proposition 4. *Suppose that history $h_m^t = \left(h^t, y_t, d_t, b_{t+1}\right)$ is equilibrium consistent. Define $q^* := \left[1 - \underline{Q}\left(0\right)\right] \times \overline{q}\left(y_t, b_{t+1}\right)$. If $Q \in \mathbb{ECD}(h_m^t)$ and $\mathbb{E}_Q\left(q_t\right) = \mu$; then, $S\left(h_m^t, \mu\right) = \left[0, \overline{\mathbb{V}ar}\left(h_m^t, \mu\right)\right]$ where $\overline{\mathbb{V}ar}\left(h_m^t, \mu\right)$ is defined as:*

[17] The equality at $q = \underline{q}\left(\cdot\right)$ follows from the strict monotonicity in q of equilibrium utility, that is given by $u\left(y_t - b_t + qb_{t+1}\right) + \beta\overline{v}\left(y_t, b_{t+1}, q\right)$. If the inequality were to be strict, then we could find a lower (certain) equilibrium consistent prices, which contradicts the definition of $\underline{q}\left(\cdot\right)$.

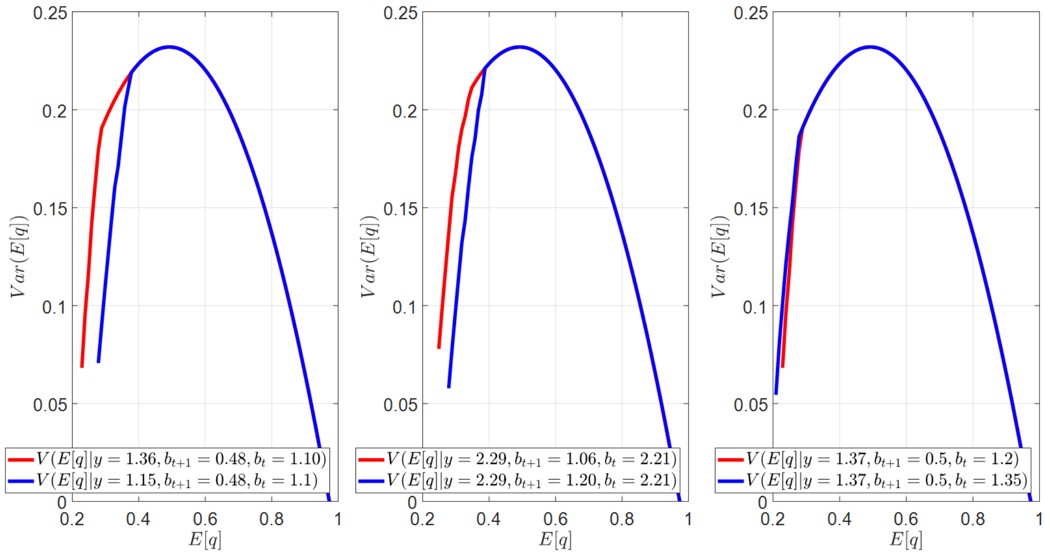

Figure 5: This figure shows $\overline{\mathbb{V}ar}\left(h_m^t, \mu\right)$ for for different levels output and for our main calibrated parameters. The left panel fixes $b_{t+1}$and b_t and gives comparative statistics with respect to y_t. The middle panel fixes $y_t$and b_t and gives comparative statistics with respect to b_{t+1}.The right panel fixes y_t and gives comparative statistics with respect to b_t.

- *If $\mu \geq q^*$, then $\overline{\mathbb{V}ar}\left(h_m^t, \mu\right) = \mu\left(\bar{q} - \mu\right)$.*
- *If $\underline{q}\left(b_t, y_t, b_{t+1}\right) \leq \mu < q^*$ then $\overline{\mathbb{V}ar}\left(h_m^t, \mu\right) = \mu\left(\bar{q} + q_\mu - \mu\right) - q_\mu \bar{q}$, where q_μ is the unique solution to $\underline{Q}\left(q_\mu\right) q_\mu + \left(1 - \underline{Q}\left(q_\mu\right)\right)\bar{q} = \mu$ and $\underline{Q}\left(q\right)$ is defined in Proposition 2.*

Proof. See Appendix A.5. □

The idea of the proof is as follows. We know that any price distribution with sunspots lies in the interval $[0, \bar{q}\left(y_t, b_{t+1}\right)]$. We start from the observation that the maximum variance is achieved with a binary distribution. For the first case, we show that the no default incentive constraint (3.3) is not binding if the expected prices are high enough; i.e., if $\mu \geq q^*$. Then, the volatility of the candidate distribution (that has a mean μ, and is binary over $\{0, \bar{q}\}$), is given by $\overline{\mathbb{V}ar}\left(h_m^t, \mu\right) = \mu\left(\bar{q} - \mu\right)$. For the second case, when $\mu < q^*$, the incentive constraint for no-default starts to be binding. The maximum variance is still achieved by a binary distribution, but this binding constraint restricts how low the price can be in the bad state. Thus, we fix q_μ such that $\Pr\left(q_\mu\right) q_\mu + \left(1 - \Pr\left(q_\mu\right)\right)\bar{q}$ is equal to μ for some probability $\Pr(q_\mu)$. In addition, we choose $\Pr(q_\mu)$ so that the incentive constraint (3.3) is binding for the candidate distribution. This probability is exactly $\underline{Q}\left(q_\mu\right)$. This is intuitive, because will make the probability of the low value as high as possible, maximizing the variance.

Figure 5 presents the bounds of the variances for the equilibrium consistent distributions given an expected value for prices. Each one of the panels and each of the two cases in each panel are different because they display different values of (b_t, y_t, b_{t+1}). First, it is clear that in the three panels that the frontier of the mean and variance has kinks. All these kinks occur when the expected price is equal to q^*. Second, note that in all of the panels, both curves are the same up to the kink of the blue line. This result occurs because q^* is a function of (b_t, y_t, b_{t+1}), which marks the kink for each one of the curves. If the expectation of prices, $\mathbb{E}(q)$, is higher than the maximum of both q^* (that is a function of the history), then the variances are identical and given by $\mu\left(\overline{q}-\mu\right)$.[18] In the right panel, the red line falls faster than the red line, because for the blue line the debt repayment is larger ($b_t = 1.35$ and $b_t = 1.2$, respectively); thus, for a given mean the variance needs to be smaller. Alternatively, in the middle panel the blue line falls faster. Because more debt is issued in the history that corresponds to the red line, for a given mean, the government tolerates higher variances of prices, without violating condition (3.3).

A General Characterization of Moments. We now formulate a simple linear program that characterizes all non-centered moments. We denote by $M_q(t)$ the moment generating function of debt prices.[19] We can characterize the maximum and minimum of the set of moments as a solution to the linear programming problem. In particular, suppose that h_m^t is an equilibrium consistent history. Then, the maximum n-th non centered moment of q solves the following linear program:

$$\overline{\mathbb{E}}(q^n \mid h_m^t) := \max_Q \frac{d^n}{dt^n}\left(\mathbb{E}^Q\left(e^{tq}\right)\right)|_{t=0}$$

subject to (3.2) and (3.3). The idea for the minimum non centered moment is analogous when we replace the max operator with the min operator. Note that this is a linear programming problem because we can interchange the expectation and the derivative. The logic of this procedure extends Propositions 2, 4, and 3.

[18] It is worth noting that for values of $\mathbb{E}(q)$ that are higher than q^*, the blue and red lines do not need to coincide. The reason why they coincide is because $\overline{q}(y_t, b_{t+1})$ is flat for both variables in the range of (y_t, b_{t+1}) in the plots.

[19] Recall that the moment generating function of the random variable q pins down all the non centered moments (a standard result in mathematical statistics); in particular:

$$\mathbb{E}(q^n) = \frac{d^n}{dt^n}\left(M_q(t)\right)|_{t=0}.$$

Comparative Statics and Stochastic Dominance. We close this subsection by providing the comparative statics over the set of distributions, $\mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$.

Corollary 2. *Suppose that h_m^t, with no default so far, is equilibrium consistent. The following comparative statistics hold: (a) The set of equilibrium price distributions $\mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$ is non-increasing (in a set order sense) with respect to b_t and if income is i.i.d, it is non-decreasing in y_t. (b) Suppose that $Q \in \mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$ and Q' is a probability distribution for equilibrium prices; i.e. $Q' \in \Delta\left(\left[0, \bar{q}\left(y_t, b_{t+1}\right)\right]\right)$. If Q' first order stochastically dominates (FOSD) Q, then $Q' \in \mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$. (c) $\underline{Q} \notin \mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$. Furthermore, for every $Q \in \mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$ it holds that Q FOSD $\underline{Q}$, and if Q' is some other lower bound, then $\underline{Q}$ FOSD Q'.*

Proof. See Appendix A.6. □

The idea of the argument follows. First, the intuition of the first part of these comparative statistics, again, stems from the revealed preference argument. If the government repaid a larger amount of debt, then the distribution of the prices that they would expect needs to shift towards higher prices. If the set does not change, then there will be a distribution that will be inconsistent with equilibrium because it will violate condition (3.3). Second, the proposition shows that once a distribution is consistent with equilibrium, any distribution that FOSD this distribution will be an equilibrium consistent distribution. This is intuitive: higher prices lead to both higher consumption and higher continuation equilibrium values for the government since both are weakly increasing in the debt price q_t. Finally, by its own definition, $\underline{Q}$ is the infimum over all possible distributions in $\mathbb{ECD}$. In addition, $\underline{Q} \notin \mathbb{ECD}\left(b_t, y_t, b_{t+1}\right)$ follows immediately from the fact that the support of $\underline{Q}$ is $\left[0, \underline{q}\left(b_t, y_t, b_{t+1}\right)\right]$.

3.5 Bounding Moments: Prices and Policies

Up to now, in Section 3, the focus has been on predictions on prices and distributions over prices given policies and output (stochastic driving variable). However, we can also obtain predictions of the *joint distribution* of policies, output, and prices, which are useful in applied settings. For example, for our model of sovereign debt, we compute the maximum volatility of prices given that the covariance of capital flows and output is negative. Note that in Proposition 4 we obtain the maximum variance given the mean expected price. In Proposition 6, we add a constraint that depends on the joint behavior of prices, policies and the stochastic driving force.

The first step is to extend Proposition 1 to a case that is useful to obtain restrictions on both prices and of policies. For this, we focus on histories, h^t, before income y_t

is realized, where the government policies $(d_t(y), b_{t+1}(y))$ are not certain. The triple $(d_t(\cdot), b_{t+1}(\cdot), Q_t(\cdot))$ is an equilibrium consistent outcome if and only if for all y the triple $(d_t(y), b_{t+1}(y), Q_t(y))$ is an equilibrium consistent outcome.

Proposition 5. *Suppose that h^t, with no default so far, is equilibrium consistent. Then, the triple $(d_t(\cdot), b_{t+1}(\cdot), Q_t(\cdot))$ where $Q_t(\cdot) \in \Delta(\mathbb{R}_+)$ is an equilibrium consistent outcome if and only if for all $y \in \mathcal{Y}$ the following hold: (a) Debt prices are SPE prices; i.e.,*

$$Q_t(y) \in \Delta\left([0, \overline{q}\left(y, b_{t+1}(y)\right)]\right) \text{ for } y : d_t(y) = 0. \tag{3.7}$$

(b) IC of the government:

$$d(y)V^A(y) + (1 - d(y)) \int_0^{\overline{q}(y, b_{t+1}(y))} \left[u\left(y - b_t + \hat{q}_t b_{t+1}(y)\right) + \beta \overline{v}\left(y, \hat{q}_t, b_{t+1}(y)\right)\right] dQ_t(\hat{q}_t; y) \geq V^A(y). \tag{3.8}$$

(c) Consistency of the default decision:

$$\mathbb{E}_y\left[1 - d_t(y) \mid h^t\right] = (1 + r) q_{t-1}. \tag{3.9}$$

Proof. See Appendix A.7. □

As in the case of Proposition 1, price distributions and policies need to be equilibrium consistent. However, now, they need to be equilibrium consistent contingent on each realization of income y_t, which is guaranteed by (3.7) and (3.8). In addition, we need to add a consistency requirement of $t-1$ prices, and t default policies, which is (3.9). The idea of the proof follows closely the one of Proposition 1.

Second, aided with the result in Proposition 5, we will characterize bounds that these conditions imply. To do so, we will obtain bounds on realized prices. Note that for low enough values of b_{t+1}, in the best equilibrium the debt prices are equal to $(1+r)^{-1}$. We define $\overline{B}(b_t, y_t)$ as the highest bond issue for which the government is indifferent between defaulting or not. By definition of $\underline{q}(\cdot)$ and $\overline{q}(\cdot)$, it holds that $\underline{q}(b_t, y_t, b_{t+1}) = \overline{q}(y_t, b_{t+1})$ when $b_{t+1} = \overline{B}(b_t, y_t)$. We denote this value of the price as $q_{\overline{B}}(b_t, y_t) := \overline{q}(y_t, \overline{B}(b_t, y_t))$. Note $q_{\overline{B}}(b_t, y_t)$ is increasing in b_t, because the worst continuation price, $\underline{q}$, is increasing in b_t. Using Proposition 3, which bounded expected prices, we know that $\mathbb{E}_\zeta(q_t \mid y) \in \left[q_{\overline{B}}(b_t, y), \frac{1}{(1+r)}\right]$. Figure 3.5 depicts the bounds on bond issuance and expected prices.

Proposition 6 characterizes the bounds on price variance $Var(q_t \mid h^t)$ given the covariance constraint given by $Cov(-b_t + b_{t+1} q_t, y_t \mid h^t) \leq -A$. For a history h^t, denote the set of income realizations where the government does not default in the best equilibrium as

$\overline{\mathcal{Y}}^{nd}(h^t)$. In particular, $\overline{\mathcal{Y}}^{nd}(h^t) := \left\{y \in \mathcal{Y} : \overline{d}(y \mid h^t) = 0\right\}$ where $\overline{d}(\cdot \mid h^t)$ is the default rule that implements the best continuation equilibrium, $\overline{v}(y_{t-1}, b_t, q_{t-1})$, after history h^t.[20] The following holds.

Proposition 6. *Suppose that the history h^t is equilibrium consistent. Then, for any equilibrium consistent outcome $(d_t(y), b_{t+1}(y), Q_t(y))$, it holds that:*

$$Var(q_t \mid h^t) \leq \min\left\{\frac{1}{4(1+r)^2}, \boldsymbol{q}_*(h^t)\left[\frac{1}{1+r} - \boldsymbol{q}_*(h^t)\right]\right\},$$

where $\boldsymbol{q}_*(\cdot)$ is the lowest equilibrium consistent expected price after history h^t. This price $\boldsymbol{q}_*(h^t)$ is defined as the solution of the program:

$$\boldsymbol{q}_*(h^t) := \min_{q(\cdot)} \mathbb{E}_y\left[q(y) \mid y_{t-1}, y \in \overline{\mathcal{Y}}^{nd}(h^t)\right], \tag{3.10}$$

subject to the constraints: (a) $q(y) \in \left[q_{\overline{B}}(b_t, y), \frac{1}{(1+r)}\right]$; and (b),

$$\mathbb{E}_y\left[q(y)\overline{B}(b_t, y)(y - \mathbb{E}(y \mid h^t)) \mid y_{t-1}, y \in \overline{\mathcal{Y}}^{nd}(h^t)\right] \geq A, \tag{3.11}$$

where $\mathbb{E}_y(y \mid h^t) := \mathbb{E}_y\left[y \mid y_{t-1}, y \in \overline{\mathcal{Y}}^{nd}(h^t)\right]$.

Proof. See Appendix A.8. □

There are two main ideas that determine the maximum variance. First, the lowest expected price after history h^t, $\boldsymbol{q}_*(h^t)$, which is given by (3.10). Note that after history h^t, the set of possible expected prices, q_t, is given by $\left[q_{\overline{B}}(b_t, y), \frac{1}{(1+r)}\right]$. The upper bound is the best equilibrium price, and the lowest bound is the lowest equilibrium expected price, which depends on the realization of output in t, which is not known at h^t. The expected price $\boldsymbol{q}_*(h^t)$ is the minimal price that we can expect, before the realization of y_t, on expectation, subject to the constraint that the price realization belongs to the set of equilibrium prices, and that we meet the covariance constraint (3.11).[21] Note that when computing expectations, we integrate over $\overline{\mathcal{Y}}^{nd}(h^t)$, because those are the realizations of output in which the country does not default in the best continuation.

Second, the maximum variance is the minimum of two terms. The first term of the min is the maximum unconstrained variance. This is the case, for example, when the

[20] Recall that this function is given by (3.1). The default rule pins down default at t. We need to define this set because the price is not defined if the government does not default. We use the default rule of the best continuation equilibrium because we would like to obtain an upper bound on the variance.

[21] The covariance is one example of a constraint that we can accommodate.

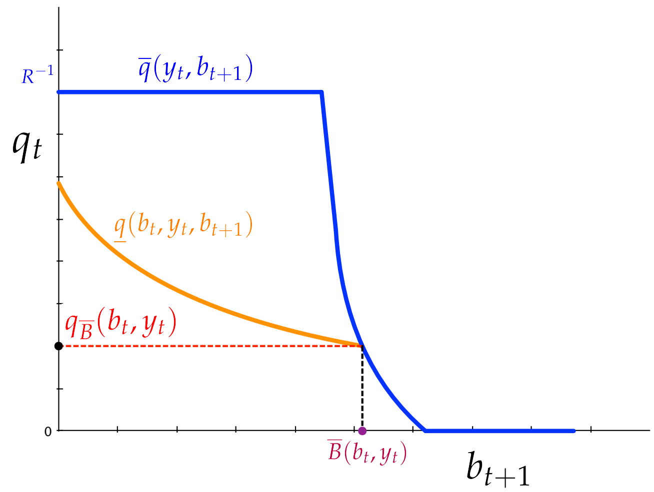

Figure 6: The x axis features different levels of debt issuance b_{t+1}. The y axis features the possible realizations of debt prices after history h^t given y_t for each value of b_{t+1}, $\overline{q}(y_t, b_{t+1})$ is the best equilibrium price function. $\overline{B}(b_t, y_t)$ is the maximum debt issuance such that the government is indifferent between defaulting and repaying. $q_{\overline{B}}(b_t, y_t)$ is the lowest (certain) equilibrium consistent prices realization with no default.

history has low debt b_t. In this case, the government can support large variance in equilibrium (and still repay the debt), so we can always find an equilibrium that rationalizes the observed history. This large variance is the one that puts probability to a price of zero and $1/(1+r)$. The second term of the min kicks in for histories in which the government enters time t with high values of debt. In this case, the government can tolerate lower variances (because otherwise it would default). In the extreme case when debt reaches a threshold, the variance due to sunspots needs to be equal to zero (but there is still fundamental variance).

4 A General Dynamic Policy Game

In this section, we show that the main result that we proved Section 3, Proposition 1, extends to a more general class of policy games. This should not be surprising. The main economic argument for Proposition 1 follows from revealed preference: what the government leaves on the table bounds its expectations regarding future play. These bounds place restrictions over outcomes or over distributions. Therefore, in this section we do three things. First, we propose a general model of a dynamic policy game in the spirit of Stokey (1991).[22] Second, for this more general setup we provide an analog of

[22] To keep notation simple and the exposition more concrete, we focus on games in which the short run players form an expectation regarding next period policy. There is a large class of models that share this

Proposition 1. Finally, we apply the general model for the case which defaults are not punished with permanent autarky.

Model. We follow the model notation in Stokey (1991). In our model, there are two types of players: an infinitely long lived player (government) and short lived agents (market) that set expectations according to a particular rule. In each period t, agents play an extensive form stage game with five sub periods $(t, \tau_i)_{i\in\{1,5\}}$. The payoff relevant states are an exogenous random shock y_t, and an endogenous state variable b_t. The timeline of the stage game is as follows:

- $\tau = \tau_1$: A publicly observable random variable $y_t \in Y \subseteq \mathbb{R}^l$ is realized, that follows a (controlled) Markov process: $y_t \sim f(y \mid y_{t-1}, b_t)$.[23]

- $\tau = \tau_2$: The long-lived player (government) chooses a control $d_t \in D \subseteq \mathbb{R}^d$ and a next period state variable $b_{t+1} \in B \subset \mathbb{R}^b$ (where both D and B are compact sets). We say that (d_t, b_{t+1}) is feasible if $(d_t, b_{t+1}) \in \Gamma(b_t, y_t)$, where $\Gamma : B \times Y \rightrightarrows D \times B$ is a non-empty, compact valued and continuous correspondence.

- $\tau = \tau_3$: A sunspot variable ζ_t is realized and distributed according to $\zeta_t \sim U[0, 1]$.

- $\tau = \tau_4$: The agents determine their expectations about future play. This process is modeled in reduced form, with the market choosing $q_t \in \mathbb{R}^k$ to satisfy:

$$q_t = \mathbb{E}_t \left\{ \sum_{s=t}^{\infty} \delta^{s-t} T(b_{s+1}, y_{s+1}, d_{s+1}, b_{s+2}) \right\},$$

  where $\delta \in (0, 1)$ and $T : B \times Y \times D \times B \to \mathbb{R}^k$ is a continuous and bounded function. The expectation is taken over future shocks $\{y_{t+s}\}_{s=1}^{\infty}$ knowing the strategy profile of the long lived player.

- $\tau = \tau_5$: the payoffs for the long lived player are realized and given by a continuous

timing. For sovereign debt, one class follows Eaton and Gersovitz (1981). For monetary policy, one class is the New Keynesian model as in Galí (2015). There are policy games that focus on alternative timings. For example, a class of games in which the decision of the long-lived player and the short-lived players occurs sequentially, but in the same period. This timing has been used mainly for monetary policy, and capital taxation. See for example Chari and Kehoe (1990). Our results can be extended to incorporate these alternative timings.

[23]Sometimes, we say that y *includes a sunspot* if $\exists \{y_t^*, z_t\}$ such that (1) $y_t^* \perp z_t$ for all t, (2) y_t^* is a controlled Markov process; i.e., $y_t^* \sim g(y_t^* \mid y_{t-1}^*, b_t)$, and (3) $z_t \sim_{i.i.d}$ Uniform $[0, 1]$.

utility function $u(b_t, y_t, d_t, b_{t+1}, q_t)$. Lifetime utility is then given by:

$$V_0 := \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(b_t, y_t, d_t, b_{t+1}, q_t) \right\},$$

where $\beta \in (0, 1)$.

Example 1. This example is exactly the one studied in Section 2. In this model, y_t is national income, $b_t \geq 0$ is the outstanding public debt to be repaid, $d_t \in \{0, 1\}$ is the default decision and $q_t = \mathbb{E}\left(\frac{1-d_{t+1}}{1+r} \mid h^{t+1}\right)$ is the risk neutral price set by lenders in equilibrium. Flow utility is given by $u(b_t, y_t, d_t, b_{t+1}, q_t) = (1 - d_t) u(y_t - b_t + q_t b_{t+1}) + d_t u(y_t)$, assuming that when the government defaults on its debt, it gets to consume its income and is banned forever from international financial markets. Note that the feasibility correspondence is given by $\Gamma(y_t, b_t, b_{t+1}, q_t) = y_t - b_t + q_t b_{t+1} \geq 0$.[24]

Example 2. A variant of our model is a model with excusable (or state contingent) debt. In such a model, the only difference is that there are no constraints on the government's ability to issue new debt after a default. Formally, the government's flow utility is now $u(b_t, y_t, d_t, b_{t+1}, q_t) = u(y_t - b_t + d_t b_t + q_t b_{t+1})$. If we allow for $d_t \in [0, 1]$ we generalize the setting to one with partial excusable defaults, as in Grossman and Huyck (1989).

Example 3. Our model also incorporates New Keynesian (NK) models of monetary policy with no endogenous state variables (e.g., Galí, 2015, Athey et al., 2005, and Waki et al., 2018). In the case of the NK model, the control variable is $d_t = \pi_t$ where π_t is inflation. Agents set inflation expectations to match future inflation, as $q_t := \pi_t^e = \mathbb{E}_t(\pi_{t+1})$. Inflation and output are related according to a forward looking Phillips curve, $x_t = \pi_t - \beta \pi_t^e$, where x_t is the output gap. In addition, let π_t^* be a random variable that gives the optimal natural level of inflation (absent an inflation gap). The random shocks are then $y_t = \pi_t^*$, and the government is assumed to minimize the loss function:

$$\mathcal{L}(\pi, \pi^e, y_t) = \frac{\chi}{2} (\pi_t - \beta \pi_t^e)^2 + \frac{1}{2} (\pi_t - y_t)^2,$$

where the first term in the loss function is the output gap. In this example, the feasibility constraint represents the fact that π_t needs to be bounded.

Consistency. As we did in Section 2, it is useful to define the best ex-post continuation payoff. We also define the set of equilibrium payoffs and the worst equilibrium payoff.

[24] Given that the market chooses after the government it can be the case that this constraint is ex-post "violated". In that case, the government has a technology available to generate resources such that the budget constraint holds; in this case the government obtains utility of $-\infty$.

We denote as $\mathcal{E}(y_-, b)$ as the *set of equilibrium payoffs*, and let $\mathcal{Q}(y_-, b) \subseteq \mathbb{R}^k$ be its projection over q.[25] The *best continuation value function* gives the maximum equilibrium value for the long lived player, if $q_t = q_-$ is realized; i.e.,

$$\overline{v}(y_-, b, q_-) := \max_{v \in \mathbb{R}} v \quad \text{s.t. } (q_-, v) \in \mathcal{E}(y_-, b). \tag{4.1}$$

By following steps that are similar to the ones used in the Appendix, Section B, we can also show that if $\mathcal{E}(y_-, b)$ is convex valued and $u(\cdot)$ is concave in q, then $\overline{v}(y_-, b, q_-)$ is also concave in q. The *max-min value* is the worst possible value that the long lived player can obtain in any SPE , going forward. Formally,

$$\underline{U}(y, b) := \max_{(d, b') \in \Gamma(b, y)} \left\{ \min_{(q, v) \in \mathcal{E}(y, b')} u\left(b, y, d, b', q\right) + \beta v \right\}. \tag{4.2}$$

In the sovereign debt model, $\underline{U}(y, b)$ is equal to $V^A(y)$.[26] Finally, we informally state a generalization of the main result presented in Section 3, Proposition 1, for the general model that we just introduced. Suppose that h_m^t is an equilibrium consistent history. Then, Q_t is an equilibrium consistent distribution if and only if: (a) SPE prices; i.e.,

$$Q_t \in \Delta\left[\mathcal{Q}(y_t, b_{t+1})\right]$$

(b) incentive compatibility for the long lived player:

$$\int_{\hat{q} \in \mathcal{Q}(y_t, b_{t+1})} \left[u(b_t, y_t, d_t, b_{t+1}, \hat{q}) + \beta \overline{v}(y_t, b_{t+1}, \hat{q})\right] dQ_t(\hat{q}) \geq \underline{U}(y_t, b_t). \tag{4.3}$$

State Contingent debt. We now study robust predictions for an extension of the Eaton and Gersovitz (1981) in which not all defaults trigger permanent autarky. In the terminology of Grossman and Huyck (1989) defaults that occur on the equilibrium path are excusable.

As we did before, the initial step is to characterize the worst equilibrium and the best continuation. First, the worst equilibrium of this alternative model is permanent autarky. Second, we denote by $\overline{v}^{SC}(y_t, b_{t+1}, q_t)$ the best equilibrium payoff function after history

[25] We can characterize this set using the concept of self-generation and enforceability in Abreu (1988); Abreu et al. (1990) and Atkeson (1991). It can be shown that if y is non-atomic and u is concave in q (for example, risk aversion of the long lived player), then $\mathcal{E}(y_-, b)$ is compact and convex valued. This is satisfied by the three xamples discussed above.

[26] There are several papers that develop the techniques to solve for the set of equilibrium payoffs following the seminal contribution of Judd et al. (2003). Following Waki et al. (2018), it can be shown that $\overline{v}(y_-, b, q_-)$ can be expressed as the unique fixed point of a contraction mapping, given $\underline{U}(y, b)$.

$\left(h^t, y_t, b_{t+1}, q_t\right)$, which is the analogue to the function $\overline{v}(y_t, b_{t+1}, q_t)$ defined in (3.1). Applying the characterization of the $\overline{v}(\cdot)$ function for the general model (see Appendix C, which builds on Waki et al. 2018) to this environment, we obtain $\overline{v}^{SC}(\cdot)$ as the unique solution to the following recursive equation:

$$\overline{v}^{SC}(y_-, b, q_-) := \max_{d(\cdot), b'(\cdot), q(\cdot)} \int \left[u\left(y - b + d(y)b + b'(y)q(y)\right) + \beta \overline{v}^{SC}(y, b'(y), q(y))\right] dF(y \mid y_-) \tag{4.4}$$

subject to

$$\begin{cases} \frac{\mathbb{E}_{y|y_-}(1-d(y))}{1+r} & = q_- \\ u\left(y - b + d(y)b + b'(y)q(y)\right) + \beta \overline{v}^{SC}(y, b'(y), q(y)) & \geq V^A(y) \text{ for all } y. \end{cases}$$

The best equilibrium price will be $\overline{q}^{SC}(y, b') = \frac{\mathbb{E}_{y'|y}\left(1 - d^{SC}(y', b')\right)}{1+r}$, where $d^{SC}\left(y', b'\right)$ is policy that solves (4.4). As in the case with $\overline{v}(\cdot)$ in our original model, is easy to show that $\overline{v}^{SC}$ is (a) strictly decreasing in b; and (b) increasing and concave in q_-. We can also show that $\overline{v}^{SC} \geq \overline{v}$.

The second step is finding the condition for equilibrium consistency. Following steps analogous to Proposition 2, we can show that Q_t is equilibrium consistent with (the equilibrium consistent history) h^t if and only if (a) $Q_t \in \Delta\left(\left[0, \overline{q}^{SC}(y_t, b_{t+1})\right]\right)$ and (b) the incentive compatibility of the government holds; i.e.,

$$\int_0^{\overline{q}^{SC}(y_t, b_{t+1})} \left[u\left(y_t - b_t + \hat{q} b_{t+1}\right) + \beta \overline{v}^{SC}(y_t, b_{t+1}, \hat{q})\right] dQ_t(\hat{q}) \geq V^A(y_t). \tag{4.5}$$

Note that the difference with respect to our previous results is a different best continuation $\overline{v}^{SC}(y_t, b_{t+1}, \hat{q})$, and a different best equilibrium price $\overline{q}^{SC}(y_t, b_{t+1})$. In this particular case of excusable defaults, the worst equilibrium payoff is again autarky. So, whether predictions are weaker in the case of excusable defaults depends on how much larger $\overline{v}^{SC}(y_t, b_{t+1}, \hat{q})$ is with respect to the one we characterized in (3.1), and how $\overline{q}^{SC}(y_t, b_{t+1})$ compares to $\overline{q}(y_t, b_{t+1})$.

5 Conclusion

Dynamic policy games have been extensively studied in macroeconomic theory to increase our understanding of how a lack of commitment restricts the outcomes that a government can achieve. One of the challenges in studying dynamic policy games is equilibrium multiplicity. Our paper acknowledges and embraces equilibrium multiplicity. For this reason, we focus on obtaining robust predictions: these are predictions that hold across all equilibria; or, in the language of Bergemann and Morris (2018), across every possible information structure. We obtain robust predictions by characterizing what we term as *equilibrium consistent outcomes*. The basis of our predictions is a revealed preference argument, which is also exploited to obtain the testable implications of equilibria in Jovanovic (1989). The idea of the revealed preference argument is that by taking a particular action, the government obtained some utility; and by doing so, incurred on some opportunity cost. This implied opportunity cost places bounds on what the government can receive in the future. Equilibrium consistency is a general principle. Even though we focus on a model of sovereign debt that follows Eaton and Gersovitz (1981), our results can be generalized to other dynamic policy games, as we show in the last section of the paper.

Appendix

A Proofs

A.1 Proposition 1

Proof. *Step 1: Necessity.*($\Longrightarrow$). *Step 1.1. Incentive compatibility of no default.* Let $\mathcal{H}(\sigma)$ be the histories on the path of a strategy profile $\sigma = (\sigma_g, q_m)$. Suppose that there is an equilibrium strategy σ such that $h_m^t \in \mathcal{H}(\sigma)$ and that there is no default so far. The fact that the government optimally decided not to default at period t implies:

$$\int_0^1 \left[u\left(y_t - b_t + q_m\left(h_m^t, \zeta_t\right) b_{t+1}\right) + \beta V^\sigma\left(h_m^t, \zeta_t\right)\right] d\zeta_t \geq u\left(y_t\right) + \beta \mathbb{E}_{y_{t+1}|y_t} V^A(y_{t+1}). \quad \text{(A.1)}$$

Step 1.2. Bounding equilibrium payoffs. We denote by $\mathcal{E}\left(y_t, b_{t+1}\right)$ the set of equilibrium payoffs of the game.[27] Since σ is an SPE, it holds that for all sunspot realizations $\zeta_t \in [0,1]$: $\left(V^\sigma\left(h_m^t, \zeta_t\right), q_m^\sigma\left(h_m^t, \zeta_t\right)\right) \in \mathcal{E}\left(y_t, b_{t+1}\right)$. The latter further implies:

a. $q_m\left(h_m^t, \zeta_t\right) \in \left[0, \overline{q}\left(y_t, b_{t+1}\right)\right]$ (i.e., it delivers equilibrium prices).

b. $\overline{v}\left(y_t, b_{t+1}, q_m\left(h_m^t, \zeta_t\right)\right) \geq V^\sigma\left(h_m^t, \zeta_t\right)$. This occurs because $\overline{v}$ is the maximum possible continuation value given the price realization $q = q_m\left(h_m^t, \zeta_t\right)$.

[27] In the Appendix Section B we define the equilibrium value correspondence and discuss how it can be computed.

Step 1.3 The distribution of prices. The price distribution implied by σ can be defined by a measure Q over measurable sets $A \subseteq \mathbb{R}_+$. More precisely:

$$Q(A) \equiv \int_0^1 \mathbf{1}\left\{q_m\left(h_m^t, \zeta_t\right) \in A\right\} d\zeta_t = \Pr\left\{\zeta_t : q_m\left(h_m^t, \zeta_t\right) \in A\right\}.$$

Note that condition (a) shows that the support of the distribution is over equilibrium prices; i.e., $Supp(Q) \subseteq [0, \overline{q}(y_t, b_{t+1})]$. *Step 1.4. The necessary condition.* By changing the integration variables in (A.1), using the definitions above, and conditions (a) and (b), we have that:

$$\int_0^{\overline{q}(y_t, b_{t+1})} \left[u\left(y_t - b_t + \hat{q} b_{t+1}\right) + \beta \overline{v}\left(y_t, b_{t+1}, \hat{q}\right)\right] dQ(\hat{q}) \geq \int_0^1 \left[u\left(y_t - b_t + q_m\left(h_m^t, \zeta_t\right) b_{t+1}\right) + \beta V^\sigma\left(h_m^t, \zeta_t\right)\right] d\zeta_t$$
$$\geq u(y_t) + \beta \mathbb{E}_{y_{t+1}|y_t} V^A(y_{t+1}),$$

which proves the desired result.

Step 2: Sufficiency ($\Longleftarrow$) Suppose that h^t is an equilibrium consistent history and that condition (3.3) is satisfied. Then, we need to construct an equilibrium strategy where at time t bond prices are distributed according to Q_t, there is no default, and bond issuance is b_{t+1} (i.e., generates h_m^t on its path). *Step 2.1. Preliminaries.* We denote by F_Q the associated cumulative probability function for Q. We denote by $\sigma^*(y_t, b_{t+1}, q)$ the strategy that achieves the continuation value $\overline{v}(y_t, b_{t+1}, q)$; i.e.,:

$$\sigma^*(y_t, b_{t+1}, q) \in \underset{\sigma \in \Sigma^*(y_t, b_{t+1})}{\operatorname{argmax}} V^\sigma\left(h^0\right) \text{ s.t. } q_0 \leq q\,.$$

As we show in the Appendix, Section B, the constraint in this problem, $q_0 \leq q$, is binding. *Step 2.2. Constructing the equilibrium strategy.* Because h^t is an equilibrium consistent history, we know there exists an equilibrium profile $\hat{\sigma} = (\hat{\sigma}_g, \hat{q}_m)$ such that $h^t \in \mathcal{H}(\hat{\sigma})$. For histories h^s successors of histories h^{t+1}, which are $h^s \succeq h^{t+1} = \left(h^t, y_t, d_t, \hat{b}_{t+1}, \zeta_t, \hat{q}_t\right)$ we define the strategy profile σ for the government as:

$$\sigma_g(h^s, y_s) := \begin{cases} \sigma^d(h^s, y_s) & \text{if } d_t = 1 \text{ or } \hat{b}_{t+1} \neq b_{t+1} \text{ or } \hat{q}_t \notin [0, \overline{q}(y_t, b_{t+1})] \\ \sigma^*(y_t, b_{t+1}, \hat{q}_t)(h^s) & \text{otherwise.} \end{cases} \tag{A.2}$$

For all $h^s \preceq h_m^t$, or $h^s \not\prec h_m^t$ or $h^s \not\succ h_m^t$, we define $\sigma_g(h^s) := \hat{\sigma}_g(h^s)$. This strategy, σ_g, prescribes the best continuation equilibrium if the government follows $(d_t = 0, b_{t+1})$ and the price that it obtains is an equilibrium price. Alternatively, if the government defaults, chooses a debt level that is different than b_{t+1}, or receives a price that is not an equilibrium price, the government will play default forever after (will be in autarky). In addition, the strategy σ_g that we just defined generates the history h_m^t on its path. Likewise, we define the strategy profile for the market. For histories $\left(h_m^t, \zeta_t\right) = \left(h^t, y_t, d_t = 0, b_{t+1}, \zeta_t\right)$, let:

$$q_m\left(h_m^t, \zeta_t\right) := F_Q^{-1}\left(\zeta_t\right), \tag{A.3}$$

where $F_Q^{-1}(\zeta) = \inf\left\{q \in \mathbb{R} : F_Q(q) \geq \zeta\right\}$ is its inverse. For $h^s \preceq h_m^t$, or $h^s \not\prec h_m^t$ or $h^s \not\succ h_m^t$, we define $q_m(h^s) := \hat{q}_m(h^s)$. For any other history, the market will choose a price of zero. *Step 2.3. Checking incentive compatibility.* Now we need to check that $d_t = 0$ and b_{t+1} is incentive compatible for the candidate strategy profile that we just constructed. Before time t, incentive compatibility comes from the fact that h_m^t is equilibrium consistent (i.e., $h_m^t \in \mathcal{H}(\sigma)$). At history h_m^t, for the candidate strategy σ, it will be optimal not to default (if we follow strategy σ for all successor nodes) if:

$$\int_0^1 \left[u\left(y_t - b_t + F_Q^{-1}\left(\zeta_t\right) b_{t+1}\right) + \beta V^\sigma\left(y_t, b_{t+1}, F_Q^{-1}\left(\zeta_t\right)\right)\right] d\zeta_t \geq u\left(y_t\right) + \beta \mathbb{E}_{y_{t+1}|y_t} V^A(y_{t+1}),$$

where $V^\sigma\left(y_t, b_{t+1}, \zeta_t\right)$ is the continuation payoff of strategy σ after $\left(y_t, b_{t+1}, \zeta_t\right)$. This condition is equivalent (if and only if) to:

$$\int_0^{\overline{q}(y_t, b_{t+1})} \left[u\left(y_t - b_t + \hat{q} b_{t+1}\right) + \beta \overline{v}\left(y_t, b_{t+1}, \hat{q}\right)\right] dQ\left(\hat{q}\right) \geq u\left(y_t\right) + \beta \mathbb{E}_{y_{t+1}|y_t} V^A(y_{t+1}), \tag{A.4}$$

where we use the fact that $F_Q^{-1}\left(\zeta_t\right) =_d \text{Uniform}\,[0,1]$, and by construction $V^\sigma\left(y_t, b_{t+1}, \zeta_t\right) = \overline{v}\left(y_t, b_{t+1}, q_t\right)$. Condition (A.4) is exactly (3.3) and thus satisfied by hypothesis. Therefore, the government does not want to deviate at time t. For any other history, because σ^d and $\sigma^*\left(y_t, b_{t+1}, \hat{q}\right)$ are subgame perfect equilibrium profiles, the government does not want to deviate. Therefore, $\sigma\left(h^s\right)$ defined in (A.2) and (A.3) is an SPE profile (since it is a Nash equilibrium at every possible history) that generates h_m^t and Q on its path. □

A.2 Corollary 1

Proof. We define $D\left(b_t, y_t, b_{t+1}, q_t\right) := u\left(y_t - b_t + q_t b_{t+1}\right) + \beta \overline{v}\left(y_t, b_{t+1}, q_t\right) - V^A\left(y_t\right)$. We can rewrite equation (3.4) as the solution to the equation $\hat{q} : D\left(b_t, y_t, b_{t+1}, \hat{q}\right) = 0$ given $\left(b_t, y_t, b_{t+1}\right)$. Note that D is strictly increasing in q when $b_{t+1} > 0$ and is a strictly decreasing function of b_t, and therefore the solution $\underline{q}$ is unique and increasing in b_t, showing (a). For (b), suppose $h = \left(h^t, y_t, b_{t+1}, d_t\right)$ with $d_t = 0$ is equilibrium consistent. Since $\overline{v}(y_t, b_{t+1}, q) \geq \mathbb{E}\left[V^A(y_{t+1}) \mid y_t\right]$, the fact that $\underline{q}$ solves the equation $D(b_t, y_t, b_{t+1}, \underline{q}) = 0$ implies $u\left(y_t - b_t + \underline{q} b_{t+1}\right) - u(y_t) = \beta\left\{\mathbb{E}\left[V^A\left(y_{t+1}\right) \mid y_t\right] - \overline{v}\left(y_t, b_{t+1}, \underline{q}\right)\right\} \leq 0$ which implies that $y_t - b_t + \underline{q}b_{t+1} \leq y_t$. Finally, for (c) note that if income is i.i.d, it holds that $V^A(y_t) = u(y_t) + \beta\frac{1}{1-\beta}\mathbb{E}\left[u(y)\right]$ and also that $\overline{v}(\cdot)$ is constant in y_t (since it does not change the expectation over next period output shocks). Therefore, the function $D(b_t, y_t, b_{t+1}, q_t)$ is differentiable with respect to y_t and $\frac{\partial D}{\partial y_t}(b_t, y_t, b_{t+1}, q_t) = u'(y_t - b_t + q_t b_{t+1}) - u'(y_t)$. Using the fact that $-b_t + \underline{q}(b_t, y_t, b_{t+1})b_{t+1} \leq 0$ and that u is a strictly concave function, at $q = \underline{q}(b_t, y_t, b_{t+1})$ we have $u'(y_t - b_t + \underline{q}(b_t, y_t, b_{t+1})b_{t+1}) > u'(y_t)$ and hence $D\left(\cdot\right)$ is a strictly increasing function of y_t. This, together with the fact that $D(\cdot)$ is increasing in q makes the solution $\hat{q} : D(b_t, y_t, b_{t+1}, \hat{q}) = 0$ decreasing in y_t , as we wanted to show. □

A.3 Proposition 2

Proof. *Step 1: Determine the upper bound for the probability of $q = 0$.* We denote by $\underline{Q}\left(\hat{q} = 0\right)$ the largest probability of a price equal to zero across all equilibrium consistent distributions. To construct $\underline{Q}\left(\hat{q} = 0\right)$ after history h^t_m, we can focus on probability distributions $\underline{Q}$ that are binary and place positive probability only on $\hat{q} = 0$ and the best equilibrium price. In this way, we relax the IC of the government as much as possible. Note that $1 - \underline{Q}\left(\hat{q} = 0\right)$ is the (lowest) probability of the best equilibrium consistent price. The IC constraint (3.3) needs to hold with equality for this distribution. Thus:

$$\underline{Q}\left(\hat{q} = 0\right)\left[u\left(y_t - b_t\right) + \beta\mathbb{E}_{y_{t+1}|y_y}V^A(y_{t+1})\right] + \left(1 - \underline{Q}\left(\hat{q} = 0\right)\right)\left[\overline{V}^{nd}\left(b_t, y_t, b_{t+1}\right)\right] = V^A\left(y_t\right).$$

This implies that:

$$\underline{Q}\left(\hat{q} = 0\right) = \frac{\Delta^{nd}\left(b_t, y_t, b_{t+1}\right)}{\Delta^{nd}\left(b_t, y_t, b_{t+1}\right) + u\left(y_t\right) - u\left(y_t - b_t\right)},$$

where $\Delta^{nd}\left(\cdot\right)$ denotes the maximum utility difference between not defaulting and defaulting (under the best equilibrium), given by $\Delta^{nd}\left(b_t, y_t, b_{t+1}\right) := V^{nd}(b_t, y_t, b_{t+1}) - V^A\left(y_t\right).$ Note further that $\underline{Q}\left(\hat{q} = 0\right)$ is bounded away from 1 from an ex-ante perspective (i.e., before the sunspot is realized, but after the government decision has been made) as long as $b_t > 0$.

Step 2: Determine the upper bound for $q = \hat{q}$. Let $p = \Pr\left(\zeta_t : q\left(\zeta_t\right) \leq \hat{q}\right)$. With a reasoning that is similar to the one in Step 1, we can conclude that by focusing on equilibria that have support $q\left(\zeta_t\right) \in \{\hat{q}, \overline{q}\left(y_t, b_{t+1}\right)\}$ we relax the IC constraint (3.3) as much possible (i.e., focus on binary distributions). Thus, we consider equilibria that assigns the best continuation equilibria when $q\left(\zeta_t\right) > \hat{q}$ (i.e $q\left(\zeta_t\right) = \overline{q}\left(y_t, b_{t+1}\right)$ and $v\left(\zeta_t\right) = \overline{\mathbb{V}}\left(y_t, b_{t+1}\right)$) and assigns $\overline{v}\left(y_-, b, \hat{q}\right)$ (the greatest continuation utility consistent with $q \leq \hat{q}$) when $q\left(\zeta_t\right) \leq \hat{q}$. The latter because $\overline{v}\left(y_-, b, \hat{q}\right)$ is increasing in $\hat{q}$. Therefore, for any such distribution, (3.3) holds: $p\left[u\left(y_t-b_t+\hat{q}b_{t+1}\right)+\beta\overline{v}\left(y_t,b_{t+1},\hat{q}\right)\right]+(1-p)\,V^{nd}\left(b_t,y_t,b_{t+1}\right)\;\geq\;V^A\left(y_t\right)$. The distribution $\underline{Q}\left(\hat{q};b_t,y_t,b_{t+1}\right)$ is defined by the equality of the previous condition. That is:

$$\underline{Q}\left(\hat{q};b_t,y_t,b_{t+1}\right)=\frac{\Delta^{nd}\left(b_t,y_t,b_{t+1}\right)}{V^A\left(y_t\right)-\left[u\left(y_t-b_t+\hat{q}b_{t+1}\right)+\beta\overline{v}\left(y_t,b_{t+1},\hat{q}\right)\right]+\Delta^{nd}\left(b_t,y_t,b_{t+1}\right)}.$$

Note that distribution $\underline{Q}\left(\hat{q};b_t,y_t,b_{t+1}\right)$ is less than 1, only when:

$$u\left(y_t-b_t+\hat{q}b_{t+1}\right)+\beta\overline{v}\left(y_t,b_{t+1},\hat{q}\right)>V^A\left(y_t\right).$$

And this happens only when $\hat{q}>\underline{q}\left(b_t,y_t,b_{t+1}\right)$, where the last inequality comes from the (alternative) characterization of $\underline{q}\left(b_t,y_t,b_{t+1}\right)$. □

A.4 Proposition 3

Proof. We already know that $\max E\left(b_t,y_t,b_{t+1}\right)=\overline{q}\left(y_t,b_{t+1}\right)$ since the degenerate distribution $\overline{Q}$ over $q=\overline{q}\left(y_t,b_{t+1}\right)$ is equilibrium consistent. In the same way, we also know that the degenerate distribution $\hat{Q}$ that assigns probability 1 to $q=\underline{q}\left(b_t,y_t,b_{t+1}\right)$ is equilibrium consistent; this distribution corresponds to a case where both investors and the government ignore the realization of the correlating device, and $\underline{q}\left(\cdot\right)$ is exactly the only price that satisfies:

$$u\left(y_t-b_t+\underline{q}\left(b_t,y_t,b_{t+1}\right)\;b_{t+1}\right)+\beta\overline{v}\left(y_t,b_{t+1},\underline{q}\left(b_t,y_t,b_{t+1}\right)\right)=V^A\left(y_t\right). \tag{A.5}$$

In the Appendix, Section B, we show that $\overline{v}\left(y_-,b,q\right)$ is a concave function in q, which together with the fact that u is strictly concave and $b_{t+1}>0$ implies that the function $H\left(q\right):=u\left(y_t-b_t+qb_{t+1}\right)+\beta\overline{v}\left(y_t,b_{t+1},q\right)$, is strictly concave in q. For any distribution $Q\in\mathbb{ECD}\left(b_t,y_t,b_{t+1}\right)$, let $\mathbb{E}_Q\left(\hat{q}\right)=\int\hat{q}dQ\left(\hat{q}\right)$. Jensen's inequality then implies that:

$$\begin{aligned}u\left(y_t-b_t+\mathbb{E}_Q\left(q\right)b_{t+1}\right)+\beta\overline{v}\left(y_t,b_{t+1},\mathbb{E}_Q\left(q\right)\right)&\underbrace{\geq}_{(1)}\int\left[u\left(y_t-b_t+\hat{q}b_{t+1}\right)+\beta\overline{v}\left(y_t,b_{t+1},\hat{q}\right)\right]dQ\left(\hat{q}\right),\\&\underbrace{\geq}_{(2)}V^A\left(y_t\right),\end{aligned}$$

with strict inequality in (1) if Q is not a degenerate distribution. Then, the definition of $\underline{q}(b_t,y_t,b_{t+1})$ implies that for any distribution $Q\in\mathbb{ECD}\left(b_t,y_t,b_{t+1}\right)$, we have that: $\mathbb{E}_Q\left(q\right)\geq\underline{q}\left(b_t,y_t,b_{t+1}\right)$. Therefore, the minimum expected value is exactly $\underline{q}\left(b_t,y_t,b_{t+1}\right)$, which is achieved uniquely at the degenerate distribution $\hat{Q}$ (because of the strict concavity of $u(\cdot)$). Finally, knowing that E is naturally a convex set, we then obtain the following:

$$E(b_t, y_t, b_{t+1}) = \left[\min_{Q \in \mathcal{Q}(b_t, y_t, b_{t+1})} \int \hat{q} dQ(\hat{q}), \max_{Q \in \mathcal{Q}(b_t, y_t, b_{t+1})} \int \hat{q} dQ(\hat{q})\right] = \left[\underline{q}(b_t, y_t, b_{t+1}), \overline{q}(b_t, y_t, b_{t+1})\right].$$

□

A.5 Proposition 4

Step 1: Bounds for General Random Variables. To show the bounds on the variance, we rely on the fact that for any random variable X with support in $[a, b] \subseteq \mathbb{R}$ and mean $\mathbb{E}(X) = \mu$, it holds that: $\mathbb{V}ar(X) \leq \mu(b + a - \mu) - ab$. Moreover, this upper bound in the variance is achieved by a binary distribution Q_μ over $\{a, b\}$, with $Q_\mu(a) = (b - \mu) / (b - a)$, and of course, $P_\mu(b) = (\mu - a) / (b - a)$.

Step 2: Are these bounds Equilibrium Consistent? It depends. Since the price realization must have support on $[0, \overline{q}(y_t, b_{t+1})]$, after history h_m^t, according to Proposition 1, we know that if Q is such that $\mathbb{E}_Q(q_t) = \mu$ then $\mathbb{V}_Q(q_t) \leq \mu(\overline{q}(y_t, b_{t+1}) - \mu)$. In addition, from step 1, we know that this bound is achieved by distribution Q_μ with support at $\{0, \overline{q}\}$, defined as $Q_\mu(0) = \frac{\overline{q} - \mu}{\overline{q}}$. However, this particular distribution may not be equilibrium consistent since it may violate the ex-ante IC for no default, condition (3.3). Whether this constraint is violated or not will depend on the particular value of $\mu \in \left[\underline{q}(b_t, y_t, b_{t+1}), \overline{q}(y_t, b_{t+1})\right]$. We define q^* as $q^* := \underline{Q}(0) \times 0 + \left(1 - \underline{Q}(0)\right) \overline{q}$ which will be key in the next steps.

Step 3. We define the function $D\left(h_m^t, .\right)$ of prices q_t as:

$$D\left(h_m^t, q_t\right) := u(y_t - b_t + q_t b_{t+1}) + \beta \overline{v}(y_t, b_{t+1}, q_t) - V^A(y_t).$$

According to condition (3.3), a distribution Q is equilibrium consistent in history h_t^m if $\mathbb{E}_Q D\left(h_m^t, q_t\right) \geq 0$. From the previous propositions we know that the function $D\left(h_m^t, q_t\right)$ is: (a) $D\left(h_m^t, 0\right) < 0$ and $D\left(h_m^t, \overline{q}\right) \geq 0$; (b) $D\left(h_m^t, q\right)$ is strictly increasing and strictly concave in q.

Step 4: Case I. $\mathbb{E}_Q(q_t) = \mu \geq q^*$. From Steps 1 and 2 we know that we can focus on distributions that put mass on $\{0, \overline{q}\}$. Note that, for a binary distribution Q, we define the function:

$$L(\mu) := \mathbb{E}_Q D\left(h_m^t, q_t; \mu\right)$$

subject to $\mathbb{E}_Q(q) = \mu$. Note that $L(\mu)$ is a strictly increasing function of μ. Thus, it is sufficient to show that $L(q^*) = 0$.

We now show that $L(q^*) = 0$. Note that, by definition of q^*, the binary distribution that yields $\mathbb{E}_Q(q) = q^*$ places probabilites $\{\underline{Q}(0), (1-\underline{Q}(0))\}$ on $\{0, \overline{q}\}$. Thus,

$$L(q^*) = \mathbb{E}_{\{\underline{Q}(0),(1-\underline{Q}(0))\}} D\left(h_m^t, q_t\right) = \underline{Q}(0)\, D\left(h_m^t, 0\right) + (1-\underline{Q}(0)) D\left(h_m^t, \overline{q}\right) = 0,$$

where the last equality follows from the characterization of $\underline{Q}(0)$ in Proposition 2.

Step 4: Case II. $\underline{q}(b_t, y_t, b_{t+1}) \leq \mathbb{E}_Q(q_t) = \mu < q^*$. Case 2. Proposal Violates IC for a Low Mean. In this case, because $L(\cdot)$ is strictly increasing we know that $L(\mu) < 0$, and we cannot use a discrete distribution with mean μ and support on $\{0, \overline{q}\}$, because it is not equilibrium consistent. However, we still know that the lower bound on the expectation of $D(h^t, q)$ can always be achieved with binary support distributions. Therefore, we look for distributions with support $\{q_\mu, \overline{q}\}$ such that:

$$\begin{cases} \lambda q_\mu + (1-\lambda)\,\overline{q} = \mu \\ \lambda D\left(h_m^t, q_\mu\right) + (1-\lambda)\, D\left(h_m^t, \overline{q}\right) = 0, \end{cases}$$

where $\lambda := \Pr\left(q_\mu\right)$. This gives a system of equations in (q_μ, λ). See that the second constraint (the no-default incentive constraint), given q_μ is the definition of the infimum distribution $\lambda = \underline{Q}\left(q_\mu\right) = \frac{D\left(h_m^t, \overline{q}\right)}{\left(D\left(h_m^t, \overline{q}\right) - D\left(h_m^t, q_\mu\right)\right)}$, given in Proposition 2. Using this into the first equation, we obtain one equation in the unknown q_μ :

$$\underline{Q}\left(q_\mu\right) q_\mu + \left(1 - \underline{Q}\left(q_\mu\right)\right) \overline{q} = \mu \iff \frac{D\left(h_m^t, \overline{q}\right) - D\left(h_m^t, q_\mu\right)}{\overline{q} - q_\mu} = \frac{D\left(h_m^t, \overline{q}\right)}{\overline{q} - \mu}. \tag{A.6}$$

Because $D\left(h_m^t, q\right)$ is increasing in q, the solution q_μ of equation A.6 is increasing in μ in the region where $\mu < q^*$.

A.6 Corollary 2

Proof. *Step 1. First Order Stochastic Dominance.* We define the function

$$U\left(Q; b_{t,} y_t, b_{t+1}\right) := \int \left\{u\left(y_t - b_t + \hat{q} b_{t+1}\right) + \beta \overline{v}\left(y_t, b_{t+1}, \hat{q}\right)\right\} dQ\left(q\right).$$

Note that this function is strictly increasing in y_t and strictly decreasing in b_t. Furthermore, the set $\mathcal{Q}(b_t, y_t, b_{t+1})$ can be rewritten as:

$$\mathcal{Q}(b_t, y_t, b_{t+1}) = \left\{ Q \in \Delta([0, \overline{q}]) : U(Q; b_{t,} y_t, b_{t+1}) \geq V^A(y_t) \right\}.$$

The function $H(q) := u(y_t - b_t + qb_{t+1}) + \beta \overline{v}(y_t, b_{t+1}, q)$ is strictly increasing in q. Therefore, if Q' FOSD Q and $Q \in \mathcal{Q}(b_t, y_t, b_{t+1})$ then $\int H(q)\, dQ' \geq \int H(q)\, dQ \geq V^A(y_t)$. *Step 2. Comparative statistics.* This follows from the fact that $U(Q; b_{t,} y_t, b_{t+1}) - V^A(y_t)$ is monotonic on y_t (when income is i.i.d.) and on b_t. *Step 3.* $\underline{Q} \notin \mathbb{ECD}(b_t, y_t, b_{t+1})$. Finally, we show that $\underline{Q}$ is not an equilibrium consistent distribution. By definition, equation 3.5 cannot be an equilibrium consistent price; this implies that the Lebesgue-Stjeljes measure associated with $\underline{Q}(\cdot)$ has the property that $\text{Supp}(Q) = \left[0, \underline{q}(b_t, y_t, b_{t+1})\right]$ and $\underline{Q}(q = 0) = p_0 > 0$, which implies that:

$$\int_0^{\overline{q}(y_t, b_{t+1})} \{u(y_t - b_t + \hat{q} b_{t+1}) + \beta \overline{v}(y_t, b_{t+1}, \hat{q})\}\, d\underline{Q}(\hat{q}) < u\left(y_t - b_t + \underline{q}(\cdot)\, b_{t+1}\right) + \beta \overline{v}\left(y_t, b_{t+1}, \underline{q}(\cdot)\right) = V^A(y_t)$$

where the last equation comes from the definition of $\underline{q}(\cdot)$ and the function $H(\hat{q})$ is strictly increasing in $\hat{q}$. □

A.7 Proposition 5

Proof. (Necessity): Suppose history h^t is equilibrium consistent. Therefore, there exist some SPE profile $\hat{\sigma} = (\hat{\sigma}_g, \hat{q}_m)$ that generated history h^t. We define the policies $(d_t(y), b_{t+1}(y)) := \left(d_t^{\hat{\sigma}_g}(h^t, y), b_{t+1}^{\hat{\sigma}_g}(h^t, y)\right)$ and the conditional price distribution defined as $Q_t(y)(A) = \Pr\left\{\zeta \in [0,1] : \hat{q}_m(h\right.$ where $A \subseteq \left[0, (1+r)^{-1}\right]$ is a measurable set of debt prices. Since σ is an SPE and it is equilibrium consistent, we know that the price q_{t-1} must be consistent with the default rule at period t; i.e., $\mathbb{E}_y(1 - d_t(y) \mid y_{t-1}) = (1+r) q_{t-1}$, which delivers condition (3.9). To show (3.7) and (3.8), we first take the shocks y, such that $d_t(y) = 0$. For this, define the triple $(d_t, b_{t+1}, Q_t) = (0, b_{t+1}(y), Q_t(y))$ and use it with Proposition 1, delivering conditions (3.7) and (3.8). The case for the shocks $d_t(y) = 1$ is immediate.

(Sufficiency): As we did in Proposition (1), since $Q_t(y)$ satisfies (3.7), we can define for all $\zeta \in [0,1]$ the price outcome $q_t(y, \zeta) := F_{Q_t(y)}^{-1}(\zeta)$ (so its distribution coincides with $Q_t(y)$). We need then to find a SPE strategy profile $\tilde{\sigma} = (\tilde{\sigma}_g, \tilde{q}_m)$: **(1)** $\left(d_t^{\tilde{\sigma}_g}(h^t, y), b_{t+1}^{\tilde{\sigma}_g}(h^t, y)\right) = (d_t(y), b_{t+1}(y))$ for all $y \in Y$ and **(2)** $\tilde{q}_m(h^t, y, d_t(y), b_{t+1}(y), \zeta_t) = q_t(y, \zeta_t)$ for all (y, ζ_t). Recall that because h^t is equilibrium consistent, there is a strategy $\hat{\sigma}$ that on its path generates h^t. For each $h^{t+1} \succ h^t$ define $\sigma^*(\cdot \mid h^{t+1}) = (\sigma_g^*, q_m^*)(\cdot \mid h^{t+1})$ to be the strategy profile that achieves the value $\overline{v}\left(y_t, b_{t+1}, q_t\right)$. We then define the following:

$$\tilde{\sigma}_g(h_g^k) := \begin{cases} (d_t(y_t), b_{t+1}(y_t)) & \text{if } h_g^k = (h^t, y_t) \\ \hat{\sigma}_g(h^k, y_k) & \text{if } k < t \text{ or } k > t : h_g^k \nsucc h^t \\ \sigma_g^*(h^k, y_k \mid h^t) & \text{for } h_g^k \succ (h^t, y_t, d_t(y_t), b_{t+1}(y_t), \zeta_t, q_t(y_t, \zeta_t)) \\ \sigma^d(h^k, y_k \mid h^t) = (1, 0) & \text{for } h_g^k \nsucc (h^t, y_t, d_t(y_t), b_{t+1}(y_t), \zeta_t, q_t(y_t, \zeta_t)) \\ & \text{and } h_g^k \text{ coincides with } h^t \text{ on first } t \text{ periods,} \end{cases}$$

$$\tilde{q}_m(h_m^k) = \begin{cases} q_t(y_t, \zeta_t) & \text{if } h_m^k = (h^t, y_t, d_t(y_t), b_{t+1}(y_t), \zeta_t) \\ \hat{q}_m(h^k, y_k, d_k, b_{k+1}, \zeta_k) & \text{if } k < t \text{ or } k > t \text{ with } h^k \nsucc h^t \\ q_m^*(h^k, y_k, d_k, b_{k+1}, \zeta_k \mid h^t) & \text{for } h_m^k \succ (h^t, y, d_t(y), b_{t+1}(y), \zeta_t) \\ q_m^d(h^k, y_k, d_k, b_{k+1}, \zeta_k \mid h^t) = 0 & \text{for } h_m^k \nsucc (h^t, y, d_t(y), b_{t+1}(y), \zeta_t) \\ & \text{and } h_m^k \text{ coincides with } h^t \text{ on first } t \text{ periods.} \end{cases}$$

See that by construction, $\tilde{\sigma}$ satisfies conditions **(1)** and **(2)**. Since the profiles $\hat{\sigma}$ (the one rationalizing h^t), σ^* and σ^d are all subgame perfect, $\tilde{\sigma}$ is a mutual best response for all histories $h \neq h^t$. Condition (3.8) shows that $\tilde{\sigma}_g$ is optimal at h_g^t. Using the definition of $q_t(\cdot)$ and condition (3.9) we have that $\tilde{q}_m(h_g^{t-1}) = q_{t-1}$ is the rational forecast given $\tilde{\sigma}_g$, finishing the proof. □

A.8 Proposition 6

Proof. *Step 1: Variance decomposition.* For a given equilibrium outcome $(d_t(y), b_{t+1}(y), Q_t(y))$ we can use the law of total variance to obtain the following:

$$Var(q_t \mid h^t) = \mathbb{E}_y \left[Var_{Q_t(y)}(q_t \mid y, h^t) \right] + Var_y \left[\mathbb{E}_{Q_t(y)}(q_t \mid y, h^t) \right]. \tag{A.7}$$

For the first term, the term between brackets is the one which we characterized in Proposition 4, and we know that $Var_{Q_t(y)}(q_t \mid y, h^t) \leq q(y)\left(R^{-1} - q(y)\right)$, where $q(y) := \mathbb{E}_{Q_t(y)}\left(q_t \mid y, h^t\right)$. For the second term, by definition of variance, note that $Var_y\left[\mathbb{E}_{Q_t(y)}(q_t \mid y, h^t)\right] = \mathbb{E}_y\left[q^2(y)\right] - \left[\mathbb{E}_y\left(q(y)\right)\right]^2$. Using both results in equation (A.7), we get that:

$$\begin{aligned} Var(q_t \mid h^t) \leq & \mathbb{E}_y \left[q(y) \left(\frac{1}{1+r} - q(y) \right) \right] + \mathbb{E}_y \left[q^2(y) \right] - \left[\mathbb{E}_y \left(q(y) \right) \right]^2 \\ & = \mathbb{E}_y \left[q(y) \right] \left(\frac{1}{(1+r)} - \mathbb{E}_y \left[q(y) \right] \right). \end{aligned} \tag{A.8}$$

Step 2: A simpler program. The problem is now reduced to look over all possible expected values $\boldsymbol{q} = \mathbb{E}_y [q(y)]$ to maximize (A.8) subject to the constraint $Cov_y(-b_t + q_t b_{t+1}, y \mid h^t) \leq -A$, for some outcome $(d_t(y), b_{t+1}(y), Q_t(y))$. To do so, we define:

$$q_* := \min_{d(\cdot), b_{t+1}(\cdot), Q_t(y)} \mathbb{E}_y \left[q(y) \mid h^t \right],$$

subject to $Cov_y(b_t - q_t b_{t+1}, y \mid h^t) \leq -A$. *Step 3: Solution to the original program.* The following holds: (a) if $q_* < (1+r)^{-1}/2$, then we can attain the unconstrained maximum, which is given by $(1+r)^{-2}/4$. (b) if $q_* \geq (1+r)^{-1}/2$, then the maximum variance is attained at q_* with a value for the variance equal to $Var(q_t \mid h^t) \leq q_* \left(\frac{1}{(1+r)} - q_* \right)$. *Step 4: Rewriting the co-variance.* For the final statement of the proposition, using the law of total co-variance, we arrive at the desired result:

$$\begin{aligned} Cov_{y,Q}(b_t - q_t b_{t+1}, y \mid h^t) &= - \mathbb{E}_{y,Q}(q_t b_{t+1}(y) y \mid h^t) + \mathbb{E}_{y,Q}(q_t b_{t+1}(y)) \mathbb{E}_y \left(y \mid h^t \right) \\ &= - \mathbb{E}_y \left(\mathbb{E}_Q \left(q_t b_{t+1}(y) y \right) \mid h^t \right) + \mathbb{E}_y \left(\mathbb{E}_Q \left(q_t b_{t+1}(y) \right) \mid h^t \right) \mathbb{E}_y \left(y \mid h^t \right) \\ &= - \mathbb{E}_y \left(b_{t+1}(y) q(y) y \mid h^t \right) + \mathbb{E}_y \left(b_{t+1}(y) q(y) \mid h^t \right) \mathbb{E}_y \left(y \mid h^t \right). \end{aligned}$$

□

B Characterization of $\overline{v}\left(y_-, b, q_-\right)$

In this Appendix we show how to compute $\overline{v}\left(y_-, b, q_-\right)$ given the equilibrium value correspondence $\mathcal{E}\left(y_-, b\right)$.[28] Note that in our model, the elements of the equilibrium value correspondence for each (y_-, b) consists in all the equilibrium pairs of utility of the government and prices of debt for investors, given an initial seed value y_- (recall that income follows a first order Markov process), and the government initially owes b bonds. The best ex-post continuation value when the income realized is y_- and b bonds are issued at price q_-, which is defined as:

$$\overline{v}\left(y_-, b, q_-\right) := \max_{\sigma \in \Sigma^*(y_-, b)} V\left(\sigma \mid y_-, b, q_-\right).$$

The function $\overline{v}\left(y_-, b, q_-\right)$ yields the highest expected utility that a government can obtain if given a realization of income y_-, they issued b bonds, and the bonds were issued at a equilibrium price q_-. Note that $\overline{v}\left(y_-, b, q_-\right)$ is the Pareto frontier in the correspondence

[28]There are several techniques that characterize $\mathcal{E}\left(y_-, b\right)$, which are now standard in the literature.

of equilibrium values:

$$\overline{v}\left(y_-, b, q_-\right) := \max\left\{v : \exists \hat{q} \geq 0 \text{ such that } \left(v, \hat{q}\right) \in \mathcal{E}\left(y_-, b\right) \text{ and } \hat{q} \leq q_-\right\}. \tag{B.1}$$

Note that we focus on a relaxed version of the problem, where we replace the equality $\hat{q} = q_-$ by the inequality $\hat{q} \leq q_-$. Proposition 7 enables us to rewrite (B.1) as a linear program. Proposition 8 enables us to compute $\overline{v}\left(y_-, b, q_-\right)$.

Proposition 7. *For all $q \in [0, \overline{q}(y_-, b)]$ the maximum continuation value $\overline{v}\left(y_-, b, q_-\right)$ solves*

$$\overline{v}\left(y_-, b, q_-\right) = \max_{d(\cdot) \in \{0,1\}^Y} \mathbb{E}_{y|y_-}\left[d\left(y\right) V^A\left(y\right) + \left[1 - d\left(y\right)\right] \overline{V}^{nd}\left(b, y\right)\right],$$

subject to:

$$q_- = \frac{\mathbb{E}_{y|y_-}\left[1 - d(y)\right]}{1 + r}. \tag{B.2}$$

Furthermore, $\overline{v}\left(y_-, b, q_-\right)$ is non-decreasing and concave in q_-.

Proof. *Step 1.1. Programming problem for an arbitrary $\tilde{v}$.* Take any $\tilde{v}$ such that:

$$\tilde{v} \in \left\{v : \exists \hat{q} \geq 0 \text{ such that } \left(v, \hat{q}\right) \in \mathcal{E}\left(y_-, b\right) \text{ and } \hat{q} \leq q_-\right\}.$$

Because $\tilde{v}$ is an equilibrium value, there exists a policy $\left(\tilde{d}\left(\cdot\right), \tilde{b}\left(\cdot\right)\right)$, such that:

$$\tilde{v} = \mathbb{E}_{y|y_-}\left[\left(1 - \tilde{d}\left(y\right)\right)\left[u(y - b + \bar{q}(y, b'(y))b'(y)) + \beta\overline{\mathbb{V}}\left(y, b'\left(y\right)\right)\right] + \tilde{d}(y)V^A(y)\right]$$

$$\left(\tilde{d}\left(y\right), \tilde{b}\left(y\right)\right) \in \arg\max_{(d(y), b'(y))} \left(1 - d\left(y\right)\right)\left[u(y - b + \bar{q}(y, b'(y))b'(y)) + \beta\overline{\mathbb{V}}\left(y, b'\left(y\right)\right)\right] + d(y)V^A(y).$$

$$\frac{\mathbb{E}_{y|y_-}\left[1 - \tilde{d}(y)\right]}{1 + r} \leq q_-.$$

Step 1.2. For a given choice of $b'(y)$, $\left(d\left(y\right), b'\left(y\right)\right)$ is an equilibrium policy if and only if, the following holds: $d(y) = 0$ implies $\overline{V}^{nd}(b, y, b'(y)) \geq V^A(y)$. *Step 1.3. The program for the largest $\tilde{v}$.* Therefore, to maximize the arbitrary $\tilde{v}$, the program is:

$$\overline{v}\left(y_-, b, q_-\right) = \max_{(d(\cdot), b'(\cdot))} \quad \mathbb{E}_{y|y_-}\left[\left(1 - d\left(y\right)\right)\overline{V}^{nd}\left(b, y, b'\left(y\right)\right) + d(y)V^A(y)\right]$$

subject to

$$\overline{V}^{nd}(b, y, b'(y)) \geq V^A(y) \text{ for all } y : d(y) = 0, \tag{B.3}$$

$$q_- \geq \frac{\mathbb{E}_{y|y_-}\left[1 - d(y)\right]}{1 + r}. \tag{B.4}$$

Step 1.4. Dropping one constraint. Note that we can relax the constraint (B.3) by choosing the optimal $b'(y)$ and we can increase the objective function. Therefore, we can substitute $\overline{V}^{nd}(b, y, b'(y))$ by $\overline{V}^{nd}(b, y)$ in (B.3). Furthermore, note that we can drop constraint (B.3), because to maximize the function you never want to violate that constraint. *Step 1.5. The price constraint is binding (B.4).* Note that if we remove the price constraint, the agent will choose the default rule to obtain price $\overline{q}(y_-, b)$ (the one associated with the best equilibrium). Thus, for any $q < \overline{q}(y_-, b)$, this constraint must be binding. Thus, the programming problem of the government is:

$$\overline{v}(y_-, b, q_-) = \max_{d(\cdot)} \quad \mathbb{E}_{y|y_-}\left[(1 - d(y))\overline{V}^{nd}(b, y) + d(y)V^A(y)\right], \tag{B.5}$$

subject to $q_- = \frac{\mathbb{E}_{y|y_-}[1-d(y)]}{1+r}$. *Step 1.6. Increasing in q_-.* Given this formulation of the problem, it is immediate that $\overline{v}(y_-, b, q_-)$ is weakly increasing in q_-. *Step 2. Concavity.* Take $q_0, q_1 \in [0, \overline{q}(y_-, b)]$. Let $d_i(y)$ with $i \in \{0, 1\}$ be one of the solutions for the program (B.5) when $q_- = q_i$ for $i \in \{0, 1\}$. Define: $d_\lambda(y) := \lambda d_0(y) + (1 - \lambda) d_1(y)$. Clearly, this might not be a feasible default policy for the program (B.5); d_λ may belong to $(0, 1)$. We solve a relaxed version of the program where $d \in [0, 1]$. Note that because the program is linear, the solution is in the boundaries and that d_λ is feasible when $q_- = q_\lambda := \lambda q_0 + (1 - \lambda) q_1$, since: $\frac{\mathbb{E}_{y|y_-}(1-d_\lambda(y))}{1+r} = \lambda q_0 + (1 - \lambda) q_1 = q_\lambda$. Therefore, the optimal continuation value at $q_- = q_\lambda$ must be greater than the objective function evaluated at d_λ. This is because the optimum will be at a corner even in the relaxed problem. We define the functional as follows:

$$G[d(\cdot)] := \mathbb{E}_{y|y_-}\left[d(y) V^A(y) + [1 - d(y)]\overline{V}^{nd}(b, y)\right].$$

Using that $G[d(\cdot)]$ is an affine functional in $d(\cdot)$, and that both $d_0(\cdot)$ and $d_1(\cdot)$ are the optimizers at q_0 and q_1, we can show that:

$$\overline{v}(y_-, b, q_\lambda) \geq G[d_\lambda(\cdot)] = \lambda \overline{v}(y_-, b, q_0) + (1 - \lambda)\overline{v}(y_-, b, q_1).$$

□

Proposition 8 solves the programming problem from Proposition 7 by reducing it to solving a problem of one equation in one unknown.

Proposition 8. *Given (y_-, b, q_-) there exists a constant $\gamma = \gamma(y_-, b, q_-)$ such that:*

$$\overline{v}(y_-, b, q_-) = \mathbb{E}_{y|y_-}\left[\underline{d}(y) V^A(y) + (1 - \underline{d}(y))\overline{V}^{nd}(b, y)\right],$$

where

$$\underline{d}\left(y\right)=0 \iff \overline{V}^{nd}\left(b,y\right)\geq V^{A}\left(y\right)+\gamma\left(y_{-},b,q_{-}\right) \textit{ for all } y\in Y \tag{B.6}$$

and γ is the (maximum) solution for the single variable equation:

$$\frac{1}{1+r}\mathbb{P}_{y|y_{-}}\left\{y\;:\;\overline{V}^{nd}\left(b,y\right)\geq V^{A}\left(y\right)+\gamma\left(y_{-},b,q_{-}\right)\right\}=q_{-}.$$

Proof. We solve a relaxed version of the programming problem in (B.5) where $d\left(y\right)\in\left[0,1\right]$. Recall that the solution will be in the corners, because we are solving a linear program. The Lagrangian is:

$$\mathcal{L}=\mathbb{E}_{y|y_{-}}\left[\left(1-d\left(y\right)\right)\overline{V}^{nd}\left(b,y\right)+d\left(y\right)V^{A}\left(y\right)\right]+\mathbb{E}_{y|y_{-}}\mu\left(y\right)\left[1-d\left(y\right)\right]\left[\overline{V}^{nd}\left(b,y\right)-V^{A}\left(y\right)\right]$$
$$+\lambda\left(q_{-}\left(1+r\right)-1+\mathbb{E}_{y|y_{-}}d\left(y\right)\right).$$

The first order condition with respect to $d\left(y\right)$ is given by:

$$\frac{\partial\mathcal{L}}{\partial\left[d\left(y\right)\right]}=\left[-\overline{V}^{nd}\left(b,y\right)+V^{d}\left(y\right)+\lambda\right]dF\left(y\mid y_{-}\right),$$

where $dF\left(y\mid y_{-}\right)$ denotes the conditional probability of state y. This implies that the optimal default rule is $\underline{d}\left(\cdot\right)$ with $\gamma:=\lambda$, and we obtain the desired result, equation (B.6).

□

C $\overline{v}\left(y_{-},b,q_{-}\right)$ with Restricted Punishments

In this section we study the case introduced in Section 4 where equilibrium values must be greater than $G\left(y_{-},b\right)$, where $G\left(y_{-},b\right)$ is mapping that provides an equilibrium value for every $\left(y_{-},b\right)$. In particular, we are interested in finding the equivalent "best equilibrium value" function when restricted to these punishments. The programming problem for $\overline{v}_{G}$ is given by:

$$\overline{v}_{G}\left(y_{-},b,q_{-}\right)=\max_{\left(v,q\right)\in\mathcal{E}\left(y_{-},b\right):v\geq G\left(y_{-},b\right)}v,$$

i.e., the best equilibrium value among all equilibrium pairs $\left(v,q\right)$ that satisfy the lower bound constraint. In the particular case where $\mathcal{E}\left(y_{-},,b\right)\geq G\left(y_{-},b\right)$, this would correspond to $\overline{v}_{G}=\overline{v}$. The programming problem for $\underline{U}_{G}\left(y,b\right)$ is given by:

$$\underline{U}_{G}\left(y,b\right):=\max_{\left(d,b'\right)\in\Gamma\left(b,y\right)}\left\{\min_{\left(q,v\right)\in\mathcal{E}\left(y,b'\right):v\geq G\left(y,b'\right)}u\left(b,y,d,b',q\right)+\beta v\right\}.$$

Generalizing the argument in Waki et al. (2018), we can link the problem of finding $\overline{v}_G$ to finding the fixed point of a contraction mapping. Namely, we will study the mapping $T : \mathcal{B}(Y \times B \times \mathbb{R}) \to \mathcal{B}(Y \times B \times \mathbb{R})$ defined as

$$T(f)(y_-, b, q_-) = \sup_{d(\cdot), b'(\cdot), q(\cdot), w(\cdot)} \mathbb{E}_y \left[u\left(b, y, d(y), b'(y), q(y)\right) + \beta w(y) \mid b, y_- \right],$$

subject to:

$$\begin{cases} u\left(b, y, d(y), b'(y), q(y)\right) + \beta w(y) \geq \underline{U}^G(y, b) & (a)\ \forall y \\ \mathbb{E}_y \left[T\left(b, y, d(y), b'(y)\right) + \delta q(y) \right] = q_- & (b) \\ G(y, b'(y)) \leq w(y) \leq f\left(y, b'(y), q(y)\right) & (c)\ \forall y \\ \left(q(y), w(y)\right) \in \mathcal{E}\left(y, b'(y)\right) & (d)\ \forall y, \end{cases}$$

where (a) is the incentive constraint, (b) is the moment condition for q_-, (c) are the bounds for continuation value, that must be above G for all values of y and below the candidate for best equilibrium value $f(y_-, b, q_-)$, and (d) that $q(\cdot), w(\cdot)$ are equilibrium payoffs. It is easy to check that T satisfies Blackwell conditions (monotonicity and discount) and is hence a contraction mapping with modulus β, and hence it has a fixed point $f^*(y_-, b, q_-)$. See that in in the sup program of T we will always have $w(y) = f\left(y, b'(y), q(y)\right)$ and then it is easy to see that the fixed point $f^*$is self-generating (see Waki et al., 2018 for an extended argument for this).

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