Optimal Debt-Maturity Management

Saki Bigio  Galo Nuño  Juan Passadore∗
UCLA  Banco de España  EIEF

October 20th 2017

PRELIMINARY AND INCOMPLETE

Abstract

We solve the problem of a government that wants to smooth financial expenses by choosing over a continuum of bonds of different maturity. The planner takes into account that adjusting debt too fast can affect prices. At the same time, it wants to insure against several sources of risk: (a) income risk, (b) interest rate (price) risk, (c) liquidity risk (prices can become more sensitive to issuance’s), and (d) the risks in the cost of default. We characterize this infinite dimensional control problem to aid the design of the debt-maturity profile in response to these forms of risk.

Keywords: Maturity, Debt Management, Open Economies

JEL classification: E32, E34, E42

∗The views expressed in this manuscript are those of the authors and do not necessarily represent the views of the European Central Bank or the Bank of Spain The authors are very grateful to Manuel Amador (discussant), Anmol Bhandhari, Alessandro Dovis, Hugo Hopenhayn, Boyan Jovanovic, Francesco Lippi, Pierre-Olivier Weill, Pierre Yared, Raquel Fernandez and the participants of SED Meetings at Edinburgh and NBER Summer Institute for helpful comments and suggestions. All remaining errors are ours.
1 Introduction

The Treasury Office of every Government or Corporation faces a large-stakes problem: how to design a strategy for the optimal issuance, pre-payment or purchases of debts of different maturities? A large literature in Economics and Finance has proposed alternative environments to study this debt management problem. The literature on sovereign debt studies a small open economy that borrows in the international debt market to smooth income shocks;\textsuperscript{1} the literature on optimal taxation studies the debt management problem as one in which the government uses debt to smooth the tax burden in a closed economy;\textsuperscript{2} corporate finance studies the debt management problem of a corporation that trades off the cost of debt dilution against the cost of issuing new debt.\textsuperscript{3} One of challenges in all of these areas is to deal with multiple assets: as we increase the number of assets the state space grows exponentially. Our main contribution is to develop a methodology to solve the debt-maturity management problem in an incomplete-markets economy where a continuum of maturities are available.

We consider the following problem: a government in a small-open economy chooses the issuance, pre-payment, or purchase, of bonds among a continuum of maturities. His financial counterpart are international investors. The planner’s objective is to smooth his expenditures, i.e., income after financial expenses. Several features complicate the planner’s problem. First, issuing debt of one particular maturity impacts the issuance price.\textsuperscript{4} Second, the planner faces several sources of risk; (a) he faces income-risk because his income is risky, (b) interest-rate (duration) risk because the international yield curve can unexpectedly change to any shape, (c) liquidity risk: the price impact of issuance’s may become more sensitive. Finally, (d) default risk: the planner can face an increase in default spreads because the process for exogenous default shocks can change. This paper develops the technology to solve this problem.

We proceed in layers by adding one trade off at a time. We start by by setting up the planner’s problem as an infinite dimensional control problem under perfect foresight.

\textsuperscript{1}This literature builds on the setting developed by Arellano (2008); Aguiar and Gopinath (2006). Examples are Chatterjee and Eyigungor (2012); Arellano and Ramanarayanan (2012) and more recently Aguiar et al. (2016). See Aguiar and Amador (2013) for a recent review of the literature on sovereign debt.

\textsuperscript{2}See for example Bhandari et al. (2016), Angeletos (2002); Buera and Nicolini (2004) and the seminal contributions of Barro (1979); Stokey et al. (1989); Aiyagari et al. (2002).

\textsuperscript{3}The seminal contribution is Leland and Toft (1996). For recent examples in this literature see He and Milbradt (2014); Chen et al. (2014).

\textsuperscript{4}A recent micro-foundation of price impact for each maturity is in Vayanos and Vila (2009). Greenwood and Vayanos (2014) test the implications of a version of the preferred habitat theory of the interest rates, finding that the supply of bonds is a predictor of the interest rates the government pays. Preferred habitat theory of the interest rates dates back at least to Culbertson (1957) and Modigliani and Sutch (1966); for a classic application to debt management see Modigliani and Sutch (1967).
The trade off for the planner is to consume and issue debt of different maturities to minimize the interest rate and adjustment cost of achieving a particular path. To solve this problem we adapt tools from the literature on Mean Fields games\(^5\) and characterize the necessary conditions for an optimal debt management and consumption policy, and develop an algorithm to compute this solution. For this case we can characterize full transitional dynamics: from any initial distribution of debt towards a steady state. One of the reasons why solving this benchmark is useful if because it provides us with the tools to analyze, numerically and analytically, unexpected-permanent shocks.

Besides the methodological contribution of solving this perfect foresight case, one insight that emerges from this problem is that it can be solved as if the government had a continuum of subordinate traders. Each trader is in charge of managing debt of a single maturity. Each trader behaves as if he was risk-neutral, but takes as given the process for the international yield curve and an “internal” discount factor. The behavior of each trader is then characterized by an individual Hamilton-Jacobi-Bellman (HJB) equation that determines the value for each type of debt. That value for the trader problem corresponds to the marginal value of debt of a particular maturity for the planner. The issuance policy for the trader problem is the optimal issuance of debt of each maturity given the assigned planner’s discount factor. The induced evolution of the debt profile then determines the planner’s disposable income, which in turn, must be consistent with the internal discount factor. As it turns out, we can solve for the HJB equation of each trader analytically, taking as given the planner’s discount factor. Hence, the only numerical requirement is to solve for a fixed-point problem in the planner’s discount factor, and as a consequence, the numerical algorithm converges in seconds.

The optimal issuance of the traders is given by a very simple formula that extends to the cases with fluctuation in interest rates and income and the case with default incentives. Issuances are given by:

\[
\iota(\tau, t) = \frac{\text{value discrepancy}}{\text{price impact}} = \frac{\psi(\tau, t) + v(\tau, t)}{\lambda(\tau, t)}.
\]

In this equation, \(\iota(\tau, t)\) is the optimal issuance at date \(t\) of a constant coupon debt whose face value matures at \(\tau\). The optimal issuance depends on the discrepancy between the international price of debt of that maturity at that date, \(\psi(\tau, t)\), and the internal valuation of the cost of debt of that maturity, \(v(\tau, t)\). Naturally, if \(\psi + v\) is positive, it is as if the trader

\(^5\)See Bensoussan et al. (2016) and for an application to Economics see Nuño and Moll (2015).
issuing that debt receives a higher amount than his net-present valuation of the cost of that debt. Without a price impact, the trader would issue an infinite amount of that debt to exploit that arbitrage. When the planner aggregates among all traders, he would notice that either he gave the traders the wrong discount factor, in which he would have to give a new instruction, or there is indeed an arbitrage. In our framework, the function $\lambda$ is a measure of the liquidity cost associated with that issuance. As long as it is a positive number, this parameter controls how quickly issuance of one type will occur.

We then introduce risk in either income, interest-rate, or liquidity. We extend the perfect foresight characterization to account for permanent and expected shocks. The importance of introducing this case is that, even though the perfect foresight case is useful in understanding the price vs cost of issuances trade off, this portfolio of debt is not determined by risk considerations. There are three features of this problem that are worth noting. First, risk does not alter the issuance margin. However, valuations for the government and the international investor are affected. Second, as opposed to the perfect foresight case, we cannot solve for the exact transitional dynamics; even in a case with Poisson shocks the problem becomes numerically intractable to solve exactly. Therefore, we study shocks that are not recurrent and we characterize exactly the risky steady state distribution of debt before the shock, and then the full transitional dynamics after the shock has hit. Third, the fact that we can model debt of different maturities introduces interesting margins. For example, when a negative income shock hits, the desire to smooth the shock by the government, added to the liquidity cost and finite maturity, produces an issuance cycle; this implies that a large portion of debt will be concentrated in a small interval of maturities, and will be due all together.

We finally turn to the case where there is risk of default. In particular, we study a setup in which there are no shocks to income, interest rates, or liquidity, but the government cannot commit to repay debt and receives an option to default with Poisson intensity. The value of the option is drawn from a distribution as in Aguiar et al. (2016). We again provide necessary conditions and an algorithm to compute the solution. To characterize the solution we study the limit of a perturbed problem in which the government has full commitment over an internal of time, as in the deterministic case. The solution of the government problem is the limit when the interval converges to zero. This solution approach provides the Markov perfect stackelberg Equilibria of the game between the government and the international investors.\footnote{Finite-dimensional Markov Perfect Stackelberg equilibria have been studied both in continuous and discrete time. See for example Başar and Olsder (1998). An example in Economics of Markov Stackelberg equilibrium is Klein et al. (2008).} As in the case with shocks to income, interest rate, or
liquidity, we study a risky steady state. We find that the planner tilts maturities towards short term debt, in line with the findings of Aguiar et al. (2016).

2 A Sovereign Borrower

Environment. Time is continuous. There is a single, freely-traded consumption good which has an international price normalized to one. The economy is endowed with a flow of $y_t$ units of the good where $\{y_t\}_{t \geq 0}$ is a continuous Markov process. The household preferences over paths for consumption $c(t)$ are given by

$$V_0 = \int_0^\infty e^{-\rho t} U(c(t)) \, dt,$$

where the instantaneous utility $U(\cdot)$ is increasing, concave, and the discount factor $\rho$ is a positive constant. Households receive transfers from the government and do not intervene in the financial market.

Government. The benevolent government (the planner) wants to maximize the utility of the representative household. To do so, the planner trades a continuum of bonds with risk-neutral competitive foreign investors. These bonds differ in their expiration dates $\tau \in (0, T]$ where $T$ is the maximum maturity. At the maturity date, the principal is repaid. The stock of outstanding bonds owed by the Government at time $t$ with a time-to-maturity $\tau$ is denoted as $f(\tau, t)$. The law of motion of the stock of maturities satisfies the Kolmogorov-Forward equation

$$\frac{\partial f}{\partial t} = \iota(\tau, t) + \frac{\partial f}{\partial \tau}, \quad (2.1)$$

where $\iota(\tau, t)$ is the new issuance of bonds of time-to-maturity $\tau$ at time $t$. The issuances $\iota(\tau, t)$ are chosen from a space of functions $\mathcal{I} : [0, T] \times (0, \infty) \to \mathbb{R}$ that satisfies technical conditions. By construction we have that $f(T^+, t) = f(0^-, t) = 0$. Finally, we let $f_0(\tau)$ be the initial stock of debt of maturity $\tau$. The budget constraint of the government is:

$$c(t) = y(t) - f(0, t) + \int_0^T [q(\tau, t, t) \iota(\tau, t) - \delta f(\tau, t)] \, d\tau, \quad (2.2)$$

---

7The stochastic process is defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

8In particular $\mathcal{I} = L^2([0, T] \times (0, \infty))$ is the space of functions on $[0, T] \times (0, \infty)$ with a square that is Lebesgue-integrable.
where \( f(0,t) \) is the repayment of the principal of the bonds at maturity, \( \int_0^T q \, d\tau \) is the amount of funds collected by issuing new debt—or spent in purchases of assets—and \( -\int_0^T \delta f \, d\tau \) is the financial expenditure repayments of the loan coupons. The problem of the government is

\[
V[f(\cdot,t)] = \max_{\{\iota(\cdot)\} \in \mathcal{I}} \mathbb{E}_t \left[ \int_t^\infty e^{-\rho(s-t)} U(c(s)) \right]
\]

(2.3)

subject to the law of motion of debt (2.1) and the budget constraint (2.2). Here \( V[f(\cdot,t)] \) is the optimal value functional, which maps the distribution \( f(\cdot,t) \) at time \( t \) into the real numbers.

**International investors.** The government sells bonds to competitive risk-neutral international investors at a price \( q(\tau,t,\iota) \). This price has two separate components:

\[
q(\tau,t,\iota) = \psi(\tau,t) + \lambda(\tau,t,\iota).
\]

The first component, \( \psi(\tau,t) \), is the valuation by the international investor of the domestic bond. This price \( \psi(\tau,t) \) is given by

\[
\psi(\tau,t) = \mathbb{E}_t \left[ e^{-\int_t^{t+\tau} \bar{r}(u) \, du} + \delta \int_t^{t+\tau} e^{-\int_t^{t+\tau} \bar{r}(u) \, du} \, d\tau \right].
\]

(2.4)

The idea is that at every \( t \) international investors can invest elsewhere in bonds of maturity \( \tau \in [0,T] \) that they buy at a price \( \psi(\tau,t) \). These bonds are priced by arbitrage given the stochastic process for the short rate \( \bar{r}(t) \). The second component, \( \lambda(\tau,t,\iota) \), represents a liquidity cost associated with issuing or purchasing \( \iota \) bonds of time-to-maturity \( \tau \). The liquidity cost \( \lambda \) is convex in \( \iota \) and the idea is that it captures the impact of many issuances of a given bond at a point in time. For the rest of the paper we work with a tractable functional form: \( \lambda(\tau,t,\iota) = -\frac{1}{2} \lambda \iota(\tau,t) \).

**Equilibrium.** We study a Markov Equilibrium with state variable \( f(\tau,t) \); it is defined as follows. A Markov equilibrium is a value functional \( V[f(\cdot,t)] \), an issuance policy \( \iota(\tau,t,f) \), bond prices \( q(\tau,t,\iota,f) \), a stock of debt \( f(\tau,t) \) and a consumption path \( c(t) \)

---

9Note that the fact that the bond obtains \( q(\tau,t,\iota) < \psi(\tau,t) \) does not mean that there is an arbitrage. One way to micro found the yield curve that the government is confronting is to follow Vayanos and Vila (2009). In this paper there is a downward sloping demand curve for bonds of the government at time \( t \) for maturity \( \tau \) coming from a preferred habitat. An alternative is to micro found \( \lambda \) as an inter mediation cost of issuance that is paid to a broker dealer.
such that: 1) Given $c(t), q(\tau, t, \iota)$ and $f(\tau, t)$ the value functional satisfies government problem (2.3) and the optimal control is; 2) Given $\iota(\tau, t, f)$ the debt stock $f(\tau, t)$ evolves according to the KPE equation (2.1); 3) Given $\iota(\tau, t, f), q(\tau, t, \iota, f), f(\tau, t)$ and $c(t)$ the budget constraint (2.2) of the government is satisfied.

3 Perfect Foresight

We start with the study of the problem of a government that faces a constant interest rate $\bar{r}$ and output $\bar{y}$, and begins with an initial condition $f(\cdot, 0)$. In particular, the planner solves $P_1$:

$$V[f(\cdot, t)] = \max_{\{\iota(\cdot)\} \in \mathcal{I}} \mathbb{E}_t \left[ \int_t^\infty e^{-\rho(s-t)}U(c(s)) \, ds \right]$$ (3.1)

subject to the law of motion of debt (2.1), the budget constraint (2.2), an initial condition $f(\cdot, 0)$, and prices of debt as given. We study this case because it reveals some interesting forces. It is also the simplest case to illustrate our solution approach. For this case we can characterize the steady state distribution of debt $f^*(\tau)$, issuances $\iota^*(\tau)$, and their determinants; an exact path for the transition from any initial distribution $f(\cdot, 0)$ to the distribution in steady state $f^*(\tau)$; and finally, the response to permanent and unexpected shocks, $f^*(\tau, \Theta) \rightarrow f^*(\tau, \Theta')$ where $\Theta, \Theta'$ are two different sets of parameters. One of the advantages of the methodology we are introducing is that even though there is a continuum of bonds, the numerical solution is simple to implement and converges in seconds.

3.1 Optimal Paths: Necessary Conditions

We now characterize the solution of $P_1$. The solution strategy for this problem involves setting up an infinite dimensional Lagrangian. We reproduce here the main ideas of the proof. First, note that the Lagrangian is given by:

$$\mathcal{L}(\iota, f) = \int_0^\infty e^{-\rho t}U \left( y(t) - f(0, t) + \int_0^T [q(t, \tau, \iota(\tau, t) - \delta f(\tau, t)] \, d\tau \right) \, dt$$

$$+ \int_0^\infty \int_0^T e^{-\rho t}j(\tau, t) \left( -\frac{\partial f}{\partial t} + \iota(\tau, t) + \frac{\partial f}{\partial \tau} \right) \, d\tau dt.$$
Lagrangian. This holds if:

\[ U'(c) \left( \frac{\partial q}{\partial \tau}(\tau, t + q(t, \tau, \iota, f) \right) = -j(\tau, t). \]

That is, the gateaux-derivative of the Lagrangian with respect to \( \iota \) has to be equal to zero. The intuition is that, by issuing or reducing debt, the government is changing the consumption of the household directly, and indirectly by changing the prices of debt. Second, there has to be no improvement up to first order with respect to the stock of debt, the state variable. Therefore, issuances equalize the marginal value of issuing debt and the marginal cost, a change in the continuation value. Taking the gateaux-derivative with respect to the state, \( f \), we obtain a PDE for the Lagrange multipliers. In particular:

\[ \rho j(\tau, t) = -U'(c(t)) \delta + \frac{\partial j}{\partial \tau}, \text{ if } \tau \in (0, T] \tag{3.2} \]

First, note that consumption is changing due to the change in debt; this is measured by the term \(-U''(c(t)) \delta\). Second, note that the variation in the stock of debt over time and maturities will imply that the multipliers associated with the stock of debt, \( j(\tau, t) \), will be changing over time. This is accounted by the terms \( \frac{\partial j}{\partial \tau}, \frac{\partial j}{\partial t}, \rho j(\tau, t) \).\(^\text{10}\) The tradeoff is clear: a change in debt will imply a change in consumption, but also, an alternative law of motion for the debt, captured by the evolution of the multipliers. It will be useful to redefine the multiplier \( j(\tau, t) \) in terms of units of consumption. Define it as:

\[ v(\tau, t) := j(\tau, t) / U'(c(t)). \]

Effectively we re-express the multipliers \( j(\tau, t) \) in terms of consumption goods. This implies that we can re-express the first order conditions with respect to \( \iota \) as

\[ \frac{\partial q}{\partial \iota}(\tau, t + q(t, \tau, \iota) = -v(\tau, t), \]

and first order conditions with respect to \( f \), the PDE equation of the multipliers (3.2), by:

\[ \left( \rho - \frac{U''(c(t)) \dot{c}(t)}{U'(c(t)) c(t)} \right) v(\tau, t) = -\delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \text{ if } \tau \in (0, T). \]

The following proposition summarizes the necessary conditions for a solution to P1.

\(^{10}\)Note that the PDE for \( j \) is the analog of the ODE of the co-state when the co-state is unidimensional. In this case, for example if the government only had access to an instantaneous bond, the PDE for the co-state would be given by \( \rho j(t) = -\delta U'(c(t)) + \frac{\partial j}{\partial t}. \)
Proposition 3.1. If a solution to $P_1$ with $e^{-\rho t} f, e^{-\rho t} i \in L^2([0, T] \times [0, \infty)), e^{-\rho t} c \in L^2[0, \infty)$, given by $\{i(\tau, t), c(t)\}_{t=0}^\infty$ exists, it satisfies the PDE

$$r(t)v(\tau, t) = -\delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \text{ if } \tau \in (0, T]$$

$$v(0, t) = -1,$$

$$\lim_{t \to \infty} e^{-\rho t} v(\tau, t) = 0$$

where $v(\tau, t)$ is the marginal value of a unit of debt with time-to-maturity $\tau$, the interest rate $r(t)$ is given by $r(t) = \rho - \frac{U''(c(t))}{U'(c(t))} \frac{\dot{c}(t)}{c(t)}$ and $e^{-\rho t} v \in L^2([0, T] \times [0, \infty))$; the optimal issuance $i(\tau, t)$ is given by

$$\frac{\partial q}{\partial t} i(\tau, t) + q(t, \tau, i) = -v(\tau, t)$$

Proof. See Appendix. \qed

3.2 Understanding Optimal Paths: Theory

In this subsection we discuss the result in Proposition 3.1 and the behavior of the benchmark when there are no adjustments costs.

Benchmark: $\bar{\lambda} = 0$. Given that we introduce adjustment costs in debt issuances, we start with the discussion for the case in which there are no adjustment costs. The proof of the necessary conditions is mechanically similar to the case with adjustment costs; however, the solution is qualitatively different. To be precise formally state it.

Proposition 3.2. If a solution to $P_1$ with $e^{-\rho t} f, e^{-\rho t} i \in L^2([0, T] \times [0, \infty)), e^{-\rho t} c \in L^2[0, \infty)$, given by $\{i(\tau, t), c(t)\}_{t=0}^\infty$ exists, it satisfies the PDE

$$r(t)v(\tau, t) = -\delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \text{ if } \tau \in (0, T]$$

$$v(0, t) = -1,$$

$$\lim_{t \to \infty} e^{-\rho t} v(\tau, t) = 0$$

where $v(\tau, t)$ is the marginal value of a unit of debt with time-to-maturity $\tau$, the interest rate $r(t)$ is given by $r(t) = \rho - \frac{U''(c(t))}{U'(c(t))} \frac{\dot{c}(t)}{c(t)}$ and $e^{-\rho t} v \in L^2([0, T] \times [0, \infty))$; valuations are such that:

$$\psi(t, \tau) = -v(\tau, t)$$
In this benchmark the solution coincides with the solution of a standard consumption savings problem with one bond. First, note that because the yield curve is arbitrage free, intuitively, the government should be indifferent regarding issuing debt of any maturity. The cost would be the same. This emerges as a feature of the solution in: \( q(t, \tau) = -v(\tau, t) \). This means that the valuation of any bond for the investors and for the government is the same. If this were not to hold, then the government would want to issue an infinite amount of one maturity. Second, note that for this equality will hold if only of \( r(t) = \bar{r} \) for all \( t \). This implies that \( \psi(\tau) = -v(\tau) \), so as long as the interest rate is constant, the valuations are equal and constant. Third, note that (for the CRRA case) we get from the definition of \( r(t) \) that \( \frac{\dot{c}(t)}{c(t)} = \frac{\rho - \bar{r}}{\sigma} \). This implies that the growth rate of consumption is constant as well. Finally, we need to pin down the initial value of consumption. This will be pinned down by the budget constraint

\[
\begin{align*}
c(t) = y(t) - f(0, t) + \int_0^T [\psi(t, \tau) \iota(\tau, t) - \delta f(\tau, t)] d\tau.
\end{align*}
\]

Note that the total amount of debt that is issued at each point in time is pinned down, \( \int_0^T \psi(t, \tau) \iota(\tau, t) \), but not on what particular maturity.

**Two Cases with \( \bar{\lambda} > 0 \).** After discussing the benchmark we now go back to the main case that we focus on this paper. It is worth noting that there are three cases in terms of \( \lambda \). The first one is \( \bar{\lambda} = 0 \), that we just discussed. The second, for a low level of lambda, is when \( \lambda \in (0, \lambda_{\text{min}}] \). This case is characterized by consumption decaying over time, and the growth rate of decay depends on lambda. This case will be detailed in the Appendix. The final case, is the one we focus in this section, \( \bar{\lambda} > \lambda_{\text{min}} \), the solution converges (in finite time) to \( \frac{\dot{c}(t)}{c(t)} = 0 \) and \( \lim_{t \to \infty} c(t) = c^{ss} \), i.e., this case is defined by a steady state with positive consumption and zero consumption growth.

**Main Case: \( \bar{\lambda} > \lambda_{\text{min}} \): Steady State.** Regarding the steady state, we can characterize valuations, issuances, and the stock of debt in closed form. Discussing the steady state equations is useful to understand the role of liquidity frictions in the model. How much the government issues of each maturity \( \tau \)? Note that because \( \frac{\dot{c}(t)}{c(t)} = 0 \), \( r(t) = \rho \). Therefore, the valuation of debt of maturity \( \tau \) for the government is given by:

\[
v^*(\tau) = -\delta \frac{1 - e^{-\rho \tau}}{\rho} - e^{-\rho \tau}.
\]
Issuances, therefore, are given by

\[ t^*(\tau) = \frac{v^*(\tau) + \psi(\tau)}{\lambda}; \]

and the stock of debt is the sum of past flows:

\[ f^*(\tau) = \int_\tau^T t^*(s) ds. \]

Recall that the valuation of international investors is given by:

\[ \psi(\tau) = \delta \frac{1 - e^{-\bar{r} \tau}}{\bar{r}} + e^{-\bar{r} \tau}. \]

The trade off is the following: the benefit of issuing debt of a particular maturity is \( \psi \), the income; the cost is the tightening of the budget constraint measured in units of consumption today \( v(\tau) \), and the issuance marginal cost \( \lambda i \).

The are four points regarding steady state debt issues that are worth noting. First, there is a tilt towards long term debt as long as the government is more impatient than international investors; that is, when \( \rho > \bar{r} \). The idea is that because the government is more impatient than the investors, as in a consumption savings problem, the government will accumulate debt. However, debt has a cost of issuance. The cost of issuance favors issuing debt less often. Note that because of the decreasing returns to scale of the issuance, the issues are not only in the long maturities. Second, note that for the case in which \( \rho = \bar{r} \) the government issues zero debt in steady state. In a standard consumption savings problem total debt in this case would be indeterminate. However, because there are issuance costs, the government does not want to issue any more debt that the one it has inherited. Third, note that as the government becomes more patient, the total debt is going down. If the government is more patient than the international investor the same logic applies, but the government accumulates assets. This is standard logic that comes from consumption savings problems. Fourth, note that the trade off in steady state is, given a stock debt, to finance it minimizing the expenditure in adjustment costs. The amount to be issued at each maturity depends on the difference in valuations of the debt by the government and the international investors. And this difference in valuations depends on their discount factor.

Figure A.1 displays the steady state of the model for the calibration with \( \rho > \bar{r} \). A steady state solves the ODE

\[ \rho v(\tau) = -\delta - \frac{\partial v}{\partial \tau}, \]
with boundary condition \( v(0) = -1 \). All quantities are expressed in percentage of the steady state output (that is equal to 1). In steady state, the country devotes 6 percent of GDP to debt service; 4.4 percent of GDP to the payment of bond principals and 1.6 percent of GDP to coupon payments. Liquidity costs, that in the current calibration are 0.3 percent of GDP, that is, about 5% of total financial expenses. New debt issuance’s are also 4.4. of GDP, since at steady state, they compensate for the payment of the principal. Consumption is 97.6 percent of GDP.

**Main Case: \( \bar{\lambda} > \lambda_{\text{min}} \): Transitional Dynamics.** Besides the steady state, the model has interesting insight regarding the transitional dynamics. First, let’s focus on the PDE for the value of each vintage. For each one of the maturities, the value evolves over time according to

\[
r(t)v(\tau,t) = -\delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}.
\]

This is the value measured in goods, of issuing a particular vintage \( \tau \). This value has a cash flow that is \(-\delta\), a discount rate \( r(t) \) that is the internal rate for all the vintages, and an evolution over time because the vintages are closer to maturity as time goes by. This value is negative and the solution is given by

\[
v(\tau,t) = -\int_{t}^{t+\tau} e^{\int_{u}^{t+\tau} r(s) ds} \delta du - e^{-r(\tau+t)}.
\]

The planner values the bonds only taking into account the internal valuation. The boundary conditions for the PDE are: \( v(\tau,t) = -1 \) and \( \lim_{t \to \infty} e^{-\rho t} v(\tau,t) = 0 \). The first one states that a bond maturing instantaneously has a marginal cost in terms of coupons of -1. All the cost comes from the principal. The second one, states that the valuation of a vintages is not exploding. If this were to be the case there would be a feasible variation that the government could exploit to increase its welfare. Second, note that the characterization provides naturally an algorithm to compute the transitional dynamics. We start from a sequence of consumption that yields an interest rate: \( r(t) = \rho + \sigma \dot{c}(t) \). Plug \( r(t) \) into value PDE:

\[
r(t)v(\tau,t) = -\delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}.
\]

Build issuances:

\[
i(\tau,t) = \frac{\psi(\tau,t) + v(\tau,t)}{\lambda}.
\]
The tradeoff regarding issuances is the same one as in steady state. Obtain \( f(t, \tau) \):

\[
f(\tau, t) = \int_{\tau}^{\min\{T, \tau+t\}} (t + \tau - s) ds + \mathbb{I}[T > t + \tau] \cdot f(\tau + t, 0)
\]

Construct a new path for consumption \( c(t) \) given by

\[
c^{out}(t) = y(t) - f(0, t) + \int_{0}^{T} [q(\tau, t) \iota(\tau, t) - \delta f(\tau, t)] d\tau
\]

And iterate until convergence. In the Numerical Appendix we discuss the numerical method to find a solution and we discuss the code.

### 3.3 Understanding Optimal Paths: The Dual

In order to state Proposition 3.1 we transformed the Lagrange multipliers \( j(\tau, t) \) into \( v(\tau, t) \) and because they solved a PDE that resembled a bond price, we referred to them as value of issuances. We now show this formally: how \( v(\tau, t) \) measures a marginal cost of issuing a unit of maturity \( \tau \) at moment \( t \). We do so by solving the dual of \( P1 \). This problem finds the lowest cost of achieving a particular path of consumption with the lowest amount of resources. Given a desired path of consumption \( c(t) \) the objective is to minimize the resources needed to achieve that path. More precisely, \( D1 \) is given by:

\[
D \left[ f(\cdot, 0) \right] = \min_{\{i(\tau, t), y(t)\} \in [0,\infty), \tau \in [0, T]} \int_{0}^{\infty} e^{-\int_{0}^{t} r(s) ds} y_t dt \text{ s.t.}
\]

\[
c(t) = y(t) - f(0, t) + \int_{0}^{T} [q(\tau, t) \iota(\tau, t) - \delta f(\tau, t)] d\tau
\]

\[
\frac{\partial f}{\partial t} = \iota(\tau, t) + \frac{\partial f}{\partial \tau}; f(\tau, 0) = f_0(\tau)
\]

\[
r(t) = \rho + \sigma \frac{\dot{c}(t)}{c(t)}
\]

Turns out that the bond prices of the traders, \( v(\tau, t) \) are the Lagrange multipliers of this optimization problem. The next Proposition establishes this result.

**Proposition 3.3.** Suppose that for a given income path \( y(t) \) and initial debt \( f_0 \) the solution to \( P1 \) is \( c^*(t), \iota^*(\tau, t), j^*(\tau, t) \). Then, \( y(t), \iota^*(\tau, t), j^*(\tau, t), \frac{j^*(\tau, t)}{v(c(t))} \) solves \( D1 \) given the path \( c^*(t) \).

**Proof.** See Appendix on Duality.
3.4 Understanding Optimal Paths: Numerical Results

Coming back to the solution of P1 in this section we want to illustrate numerically two main forces that drive the solution: consumption smoothing versus the smoothing of adjustment costs. We will study a permanent and unexpected shock to output and the interest rate that reverts to the long run initial value of them. In particular, we study an AR(1) income path for $y(t)$ to illustrate the issuance cycle and an AR(1) path for the short rate $r(t)$. Transitional dynamics reveal two forces: issuance cycles and consumption vs. price smoothing.

**A Shock to Output.** In Figures A.2 and A.3 we analyze the response of issuances, consumption and total debt from a shock to output of 5% that reverts to the long run steady state as an AR(1) process. The main take out is that to smooth the shock the government increases issuances on impact, and this will generate a wave of payments concentrated in $T$ years. Upon impact, we observe two things. We see an immediate increase of issuances and a pronounced increase on impact of the internal discount factor. The liquidity cost prevents a perfect smoothing of consumption. This is why the internal discount factors jump. Also, there is a cycle of payments. As the initial vintage of borrowings matures, and it is particularly pronounced for long-term bonds, consumption growth slows down, but then accelerates again as the wave passes. This is an interesting phenomenon because it suggests that in presence of liquidity costs, we should expect waves of debt refinancing.

**A Shock to the Interest Rates.** In Figures A.5 and A.6 we analyze the response of issuances, consumption and total debt from a shock to the interest rate, that goes to zero, and returns to the long run steady state as an AR(1) process. We compare the responses when $\sigma = 2$ and $\sigma = 0$. When the IES is not infinite, the model shows that when rates are unusually low, the country increases it’s borrowing. This is captured by a spike in consumption beyond it’s steady state level. Then, as rates begin to increase, the issuance rate declines. Eventually, there’s a period low consumption were debt is being repaid. The reason for this repayment phase is the liquidity cost. As rates return to normality, while the stock is higher due to the past issuances, the country is making higher interest and principal payments, which take consumption to a lower level than at steady state. As the debt is repaid and issues return to steady state, consumption converges back to steady state. Turning on a consumption smoothing motive tampers this effect. There’s a trade-off between exploiting the low interest rate environment and smoothing consumption.
Take Away Deterministic Dynamics. First, Steady-State: preference for long-term debt. Second, the importance of maximal maturity $T$: produces an issuance cycle. Third, the role of the IES: Adjustment cost vs. consumption smoothing.

4 Risky Steady State

We now study the role of risk in the dynamics of government debt. The goal is to study how the small-open economy should behave while expecting a shock. In particular, the country is expecting a jump to a new path of output or short-term rates. The new path converges to the same or a different steady state. To shorten the exposition, our derivation here focuses on the case of a sudden drop in output; however, the same argument applies to a change in the short interest rate. Thus, for now, we assume that the endowment process follows a two-state Markov process with values $\{y^H, y^L\}$ where state $y^L$ is absorbing. If the economy is in state $y^H$, it may jump to state $y^L$ at a rate $\phi$. This might be interpreted as if the economy is initially in a state $y^H$ (“normal times”) and may receive an aggregate shock that permanently reduces its output to $y^L$. The problem of the government, $P_2$, in this case is:

$$V \left[ f \left( \cdot , t \right) , y^H \right] = \max_{\{i(\cdot)\} \in \mathcal{I}} \mathbb{E}_t \left[ \int_t^{\tau^C} e^{-\rho(s-t)} U(c(s)) \, ds \right. $$

$$\left. + e^{-\rho(\tau^C-t)} V \left[ f \left( \cdot , \tau^C \right) , y^L \right] \mid y_t = y^H \right]$$

before the shock hits and

$$V \left[ f \left( \cdot , t \right) , y^L \right] = \max_{\{i(\cdot)\} \in \mathcal{I}} \mathbb{E}_t \left[ \int_t^{\infty} e^{-\rho(s-t)} U(c(s)) \, ds \mid y_t = y^L \right] ,$$

after the shock hits, subject to the law of motion of the debt distribution, the budget constraint and the price of debt. The price of debt will not depend on the state, because there is no risk of default. Here $\tau^C$ is the first arrival time of the aggregate shock, and it is distributed according to an exponential random variable with parameter $\phi$. The dynamics of $i \left( f \left( \cdot , t \right) , y^L \right)$ are given by the equations described the previous Section as the economy converges to the steady state with endowment $y^L$ and remains there.
4.1 Optimal Paths: Theory

We provide an intuition for the proof of the necessary conditions for an optimal path. Note that from the principle of optimality we can show:

\[
\rho V[f(\cdot,t)] = \max_{i(\cdot,t)} \mathcal{U}(c(t)) + \int_0^T \delta V \frac{\partial f(\cdot,t)}{\partial t} d\tau + \phi [\hat{V}[f(\cdot,t)] - V[f(\cdot,t)]],
\]

where \(\hat{V}[f(\cdot,t)] := V[f(\cdot,t), y^L] \) and \(V[f(\cdot,t)] := V[f(\cdot,t), y^H]\). The flow value of utility, the left hand side, has to be equal to the static utility of consumption plus the expected change in the continuation value. The only non-standard term is coming from the fact that the distribution is changing not only because of the shock, but also because the distribution of maturities can be changing over time; the second term on the right hand side of the HJB. The necessary conditions come from taking first order conditions in the HJB. First, note that from the first order condition with respect to issuance is given by:

\[
\mathcal{U}'(c) \left( \frac{\partial q}{\partial t} i(\tau,t) + q(\tau,t,i) \right) = -\frac{\delta V}{\delta f}.
\]

and if we define \(j(\tau,t) := \frac{\delta V}{\delta f}\), then the first order condition results in

\[
\mathcal{U}'(c) \left( \frac{\partial q}{\partial t} i(\tau,t) + q(\tau,t,i) \right) = -j(\tau,t).
\]

So, as in the perfect foresight case, the marginal cost of issuances, that is the sum of the multiplier and the adjustment cost, equalizes to the the marginal revenue benefit. Second, from the first order conditions with respect to \(f\):

\[
\begin{align*}
(\rho + \phi) j(\tau,t) &= \mathcal{U}'(c(\cdot)) (-\delta) + \frac{\partial j}{\partial \tau} - \frac{\partial j}{\partial t} + \phi \hat{j}(\tau,t), \text{ if } \tau \in (0,T) \\
j(0,t) &= -\mathcal{U}'(c(\cdot)), \text{ if } \tau = 0.
\end{align*}
\]

If we define the variables

\[
\begin{align*}
v(\tau,t) &= j(\tau,t) / \mathcal{U}'(c^H(t)), \\
\hat{v}(\tau,t) &= \hat{j}(\tau,t) / \mathcal{U}'(c^L(t)),
\end{align*}
\]

we can spell out our main result where \(c^H(t)\) is consumption before the shock hits, and \(c^L(t)\) is consumption after the shock hits. The following proposition states the main re-
result of this section. Note the terms we had before there is an additional one that takes into account the fact that with probability $\phi$ the shock to output arrives; this is the usual correction for risk of a Poisson event. We are now ready to state the main result of the section.

**Proposition 4.1.** If a solution to $P2$ with $e^{-\rho t} f$, $e^{-\rho t} l \in L^2([0, T] \times [0, \infty)), e^{-\rho t} c \in L^2[0, \infty)$ given by \( \{i(\tau, t), c(t)\}_{t=0}^{\infty} \) exists, it satisfies the PDE

\[
\left( \rho - U''(c^H(t)) \frac{dc^H}{dt} \right) v(\tau, t) = -\delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau} + \phi \hat{\phi}(\tau, t) \frac{U'(c^L(t))}{U'(c^H(t))}, \text{ if } \tau \in (0, \infty),
\]

\[
v(0, t) = -1, \text{ if } \tau = 0,
\]

\[
\lim_{t \to \infty} e^{-\rho t} v(\tau, t) = 0,
\]

the optimal issuance $i(\tau, t)$ is given by

\[
\frac{\partial q}{\partial \tau} i(\tau, t) + q(\tau, t, i) = -v(\tau, t),
\]

before the shock hits, and after the shock hits is given by

\[
\hat{\pi}(\tau, t) = -\delta + \frac{\partial \hat{\pi}}{\partial t} - \frac{\partial \hat{\pi}}{\partial \tau}, \text{ if } \tau \in (0, T]
\]

\[
\hat{\pi}(0, t) = -1,
\]

\[
\lim_{t \to \infty} e^{-\rho t} \hat{\pi}(\tau, t) = 0
\]

where $\hat{\pi}(\tau, t)$ is the marginal value of a unit of debt with time-to-maturity $\tau$, the interest rate $r(t)$ is given by $r(t) = \rho - \frac{U''(c(t)) c^L(t)}{U'(c(t)) c^H(t)}$ and $e^{-\rho t} \hat{\pi} \in L^2([0, T] \times [0, \infty))$; the optimal issuance $i(\tau, t)$ is then given by

\[
\frac{\partial q}{\partial \tau} i(\tau, t) + q(\tau, t, i) = -\hat{\pi}(\tau, t).
\]

**Proof.** See Appendix.

\[\square\]

### 4.2 Understanding Optimal Paths: Theory

The solution to Proposition 4.1 is a fixed-point in a suitable space of functions with domain $[0, T] \times [0, \infty)$. To see this, notice that the value function $v(\tau, t)$ before the shock arrival depends on the continuation value $\hat{\pi}(\tau, t)$, which in turn is a function of the value of the distribution $f(\tau, t)$ at the (unknown) time of the shock arrival. As the distribution depends on the issuances $i(\tau, t)$ and hence on the value function itself $v(\tau, t)$. The nu-
A numerical solution to this fixed point problem would be extremely complex, as it requires iterating over spaces of functions. However, in a particular case of interest (the risky-steady state) the system can be analyzed in a tractable way, as we describe next. More specifically, given an initial distribution of debt \( f_0 \), the solution involves the government choosing a distribution of debt at each point in time just in case the shock hits. In turn, this choice depends on the continuation values, so it implies a fixed point in a sequence of distributions.

**Benchmark:** \( \lambda = 0 \). We start again with the case in which there are no adjustment costs. We state the Proposition precisely below.

**Proposition 4.2.** If a solution to \( P2 \) with \( e^{-\rho t} f \), \( e^{-\rho t} l \in L^2 ([0, T] \times [0, \infty)) \), \( e^{-\rho t} c \in L^2 [0, \infty) \) given by \( \{ l(\tau, t), c(t) \}_{t=0}^{\infty} \) exists, it satisfies the PDE

\[
\left( \rho - \frac{U''(c^H(t))}{U'(c^H(t))} \frac{dc^H}{dt} \right) v(\tau, t) = -\delta + \frac{\partial v}{\partial \tau} - \frac{\partial v}{\partial \tau} + \phi \hat{v}(\tau, t) \frac{U'(c^L(t))}{U'(c^H(t))}, \text{ if } \tau \in (0, \infty),
\]

\[
v(0, t) = -1, \text{ if } \tau = 0,
\]

\[
\lim_{t \to \infty} e^{-\rho t} \hat{v}(\tau, t) = 0,
\]

the optimal issuance \( i(\tau, t) \) is given by

\[
q(\tau, t) = -v(\tau, t),
\]

before the shock hits, and after the shock hits is given by

\[
r(t) \hat{v}(\tau, t) = -\delta + \frac{\partial \hat{v}}{\partial \tau} - \frac{\partial \hat{v}}{\partial \tau}, \text{ if } \tau \in (0, T]
\]

\[
\hat{v}(0, t) = -1,
\]

\[
\lim_{t \to \infty} e^{-\rho t} \hat{v}(\tau, t) = 0
\]

where \( \hat{v}(\tau, t) \) is the marginal value of a unit of debt with time-to-maturity \( \tau \), the interest rate \( r(t) \) is given by \( r(t) = \rho - \frac{U''(c(t))}{U'(c(t))} \frac{c(t)}{c(t)} \) and \( e^{-\rho t} \hat{v} \in L^2 ([0, T] \times [0, \infty)) \); the optimal issuance \( i(\tau, t) \) is then given by

\[
q(\tau, t) = -\hat{v}(\tau, t).
\]

The characterization of an optimal path in this case, again follows the one for the case of one bond. First, note that because the yield curve is arbitrage free and there is no risk of default, investors will be indifferent among bonds of different maturities. The shock is
to output. At the same time, the government is also indifferent. This emerges as a feature of the solution in: \( q(t, \tau) = -v(\tau, t) \). This means that the valuation of any bond for the investors and for the government is the same. Second, note that for this equality to hold if only of \( r(t) = \bar{r} \) for all \( t \). This implies that \( \psi(\tau) = -v(\tau) \) and \( \psi(\tau) = -\hat{\psi}(\tau) \) so as long as the interest rate is constant, the valuations are equal and constant. Third, note that from the definition of \( c(t) = \rho - \bar{r} \) and this implies that \( \hat{\psi}(\tau) = -v(\tau) \) and \( \hat{\psi}(\tau) = -\hat{v}(\tau) \) so as long as the interest rate is constant, the valuations are equal and constant. Fourth, finally, we need to pin down the initial value of consumption.

**Risky Steady State.** We will characterize debt, issuances and consumption, for the case in which the economy is waiting for the shock but the shock has not arrived yet; that is, in histories where the shock has not occurred yet and the economy already converged to a steady state. In this case the solution of the system is composed of the following equations:

\[
(\rho + \phi) v(\tau) = -\delta - \frac{\partial v}{\partial \tau} + \left[ \phi \frac{U'(c^L(0))}{U'(c^H(0))} \right] \hat{\psi}(\tau, 0), \text{ if } \tau \in (0, \infty)
\]

\[
v(0) = -1 \text{ if } \tau = 0
\]

\[
\left(\rho - \frac{U'(c^L(t))}{U'(c^L(t))} \frac{dc^L}{dt}\right) \hat{\psi}(\tau, t) = -\delta + \frac{\partial \hat{\psi}}{\partial t} - \frac{\partial \hat{\psi}}{\partial \tau}, \text{ if } \tau \in (0, \infty), \hat{\psi}(0, t) = -1, \text{ if } \tau = 0,
\]

\[
c^H = y^H - f^H(0) + \int_0^T \left[ q\left(\tau, i^H(\tau)\right) i^H(\tau) - \delta f^H(\tau) \right] d\tau,
\]

\[
c^L(t) = y^L - f^L(0, t) + \int_0^T \left[ q\left(\tau, i^L(\tau, t)\right) i^L(\tau, t) - \delta f^L(\tau, t) \right] d\tau,
\]

\[
\frac{\partial q}{\partial t} i^H(\tau, t) + \psi\left(\tau, i^H(\tau)\right) = -v^H(\tau),
\]

\[
\frac{\partial q}{\partial t} i^L(\tau, t) + \psi\left(\tau, i^L(\tau, t)\right) = -v^L(\tau, t),
\]

\[
0 = i^H(\tau) + \frac{\partial f^H}{\partial \tau}
\]

\[
\frac{\partial f^L}{\partial t} = i^L(\tau, t) + \frac{\partial f^L}{\partial \tau},
\]

\[
f^L(\cdot, 0) = f^H(\cdot).
\]

We study this case because we can obtain an exact solution for the transitional dynamics. The case in which the shock can arrive at any moment involves computing the fixed point in a sequence of distributions and is numerically unfeasible without an approximation. An analogous challenge appears in models of incomplete markets and one proposed solution is an approximation as in Krusell and Smith (1998).
4.3 Optimal Paths: Risky Steady State Numerical Results

Next, we present two numerical exercises for the general case. In the first exercises, output is expected to drop by 5% on impact, and then a recovery back to steady state. **Expecting a 5% \( y(t) \) drop.** First, we compare the risky steady state (red), with the steady state (blue). There are two patterns to be dissected from Figures A.7 and A.8. On one hand, the presence of income risk produces an overall decline in issuances and debt outstanding of all maturities. On the other hand, we see a relative decline in shorter maturities. The reason for the decline in overall borrowing is the presence of risk, which captured by the ratio of marginal utilities:

\[
\frac{\phi U'(c_L(0))}{U'(c_H)}
\]

present in the valuation formulas. This ratio tells us how costly an outflow of payments is once the shock is realized. This penalizes all bonds, and in the valuation equations the effect is analogous to an expected increase in coupon payouts. The second observation is that the decline is more pronounced on short term assets. It’s better to explain the logic in the next exercise.

**Shock 4% to 8% on impact, \( \rho = 6\% \).** In this experiment, depicted in Figures A.9 and A.10, we present an expected shock where short term rates are expected to increase suddenly, to 8%. This is a rate above the discount factor of the small economy. As in the previous exercise, the effect of this source of risk is to shrink the issuances at all maturities. However, the exercise results in a more extreme reversal of positions. For very short-term bonds, the country actually begins buying back and eventually accumulating short term bonds. This means that the way the country reacts to risk is by issuing long-term debt but holding short term assets. Again, all of the logic is captured by the valuation equations. A long-term bond will be long lived. With a high probability, it will experience a cycle that begins with a positive spread between \( r(t) \) and \( \bar{r}(t) \), but then, when the shock is realized, with a converse relationship. When international rates are high, the country is worse off by holding those assets, because paying out coupons when discounts are high is not desirable. However, there are two other phases where the country spread is positive. This makes long-term debt desirable, despite the fact that with a high probability there will be a period where the debt will be particularly costly because marginal utility is high. The principle, on the other hand, will be likely to be repaid once marginal utility is close to steady state again. Short-term debt is different. Before the shock is realized, the short-term bond yields a benefit because \( r_{ss} - \bar{r}_{ss} > 0 \). However, being a short-term bond, it is likely to experience a period of high internal discounting. The chances of entering this period are the same as for the long-term bond. However, the short bond doesn’t have the
length to cover the reversal of the cycle. Furthermore, it’s principle is likely to be paid in
periods of high marginal utility. This makes the short-term bond be valuable as an asset.

Both exercises have the same forces behind. However, the calibration of interest-rate
risks are strong enough to reverse the positions.

5 Defaultable Debt

We now introduce the option to default for the government. The nature of the problem
changes since this interaction is a game between the government and the international
investors: the prices, as opposed to the first two cases, will depend on the actions of the
government, and these actions will in turn depend on prices. Our strategy to solve this
strategic interaction will be to spell out a problem where the government can only default
in fixed moment in time, being precise about the timing of actions, and then take the limit
when the time interval between default options goes to zero. We start this section by
discussing how the setup of the first sections of the paper is modified to include the option
to default. Then we state and show the main result: a characterization of the consumption
and debt policies when there is risk of default. We finally study a risky steady state and
numerically solve the model.

Government. The setup is analogous to the one in the main section but now the gov-
ernment has the option to default on debt. This option arrives as a Poisson process with
intensity $\theta$. Once the government receives the option to default it draws a value $V^D$ from
$[V_L, V_U]$ according to some distribution $\Phi$. It decides to default or not by comparing this
value with the value of repayment. When it defaults, it defaults on all loans. We denote
by $\tau^D$ the time to default. The latter is a stopping time with respect to the filtration $\{\mathcal{F}_t\}$.
If we define $\tau^n$ as the $n$–th arrival time of the Poisson shock that grants the government
the options to default, then

$$\tau^D := \min\left\{\tau^n, V^D > V[f(\cdot,t + \tau^n)]\right\}.$$  

International Investors. The government sells bonds to competitive risk-neutral foreign
investors that can invest elsewhere at the risk-free real rate $\bar{r}(\tau, t)$. Notice that we allow
for a (stochastic) time-varying (risk free) yield curve where $\bar{r}(\tau, t)$ is the discount factor
for a cash flow of maturity $\tau$ at time $t$. The price of each bond $q$ is, again, composed by
two separate components:

\[ q(\tau, t, \iota) = \psi(\tau, t) + \lambda(\tau, t, \iota), \]

where \( \psi(\tau, t) \) is the net-of-liquidity cost price of a bond of time-to-maturity \( \tau \) in the international market and \( \lambda(\tau, t, \iota) \) is the liquidity cost associated with issuing or purchasing \( \iota \) bonds of time-to-maturity \( \tau \). The value of the defaultable bonds \( \psi(\tau, t) \) is given by

\[
\psi(\tau, t) = \mathbb{E}_t \left[ \int_t^{t+\min\{\tau^D, \tau\}} e^{-\int_s^t \bar{r}(s,u)du} \delta ds + 1 \left( \tau < \tau^D \right) e^{-\int_t^{t+\tau} \bar{r}(\tau,u)du} \right]. \tag{5.1}
\]

The first term is the coupon payment received by investors up until the government defaults on debt, or up until maturity; that is why the integration limit is \( t + \min\{\tau^D, \tau\} \). The second term is the value of the principal that investors will receive if the government does not default before the maturity of the bond; that is, if \( \tau^D \) the time to default is larger than the time to maturity.

**Government Problem.** The problem of the government, PD, is

\[
V[f(\cdot, t)] = \max_{\{i(\cdot)\} \in \mathcal{I}} \mathbb{E}_t \left[ \int_t^{t+\tau^D} e^{-\rho(s-t)U(c(s))}ds + e^{-\rho \tau^D} V^{D} \right], \tag{5.2}
\]

subject to the law of motion of debt (2.1), the budget constraint (2.2), and the prices (5.1). Here, again, \( V[f(\cdot, t)] \) is the optimal value functional, which maps the distribution \( f(\cdot, t) \) at time \( t \) into the real numbers.\(^{11}\)

5.1 Optimal Paths: Theory

We focus on the case that the government faces a constant interest rate \( \bar{r} \) and output \( \bar{y} \). To solve PD we proceed in two steps. First, we setup a discrete commitment problem. In this problem the government can default but only at \( \Delta \) time intervals. Second, we take the continuous time limit of this problem when \( \Delta \to 0 \), that is, the government can default at any moment in time.

**Discrete Commitment Problem.** In this problem \( \Delta \) is defined as an arbitrary time step such that the government receives the option to default at the end of the interval \( (t, t + \Delta] \).

\(^{11}\)There set \( \mathcal{I} \) is given by \( \mathcal{I} = L^2 \left([0, T] \times [0, \infty)\right) \) the space of functions on \([0, T] \times [0, \infty)\) with a square that is Lebesgue-integrable.
This implies that the riskless bond price results in:

\[
\psi(\tau, t) = \int_t^{\min(\Delta, \tau)} e^{-r(s-t)} \delta ds + 1_{\{\tau \leq \Delta\}} e^{-\bar{r}\tau} + 1_{\{\Delta < \tau\}} e^{-\bar{r}\Delta} \left[ P(\tau^1 \leq \Delta) \Phi(V[f(\cdot, t + \Delta)]) \psi(\tau, t + \Delta) + P(\tau^1 > \Delta) \psi(\tau, t + \Delta) \right]
\]

(5.3)

where and \( \tau^1 \) is the first arrival time of the option to default, characterized by an exponential random variable with parameter \( \theta \). Note that the first two components are deterministic. Applying the Feynman-Kac formula to the bond pricing equation (5.3), where the stopping time is the time to default, yields the following PDE for the price of the bond:

\[
\bar{r} \psi(\tau, t) = \delta + \frac{\partial \psi}{\partial t} - \frac{\partial \psi}{\partial \tau}, \text{ if } \tau \in (0, T)
\]

\[
\psi(0, t) = 1, \text{ if } \tau = 0,
\]

\[
\psi(\tau, t + \Delta) = P(\tau^1 \leq \Delta) \Phi(V(t)) \psi(\tau, t + \Delta) + P(\tau^1 > \Delta) \psi(\tau, t + \Delta), \tau \in (0, T)
\]

(5.4)

where

\[
\Gamma(x) : = \int_x^{\infty} \xi d\Phi(\xi).
\]

The problem of the government is then given by:

\[
V(t) : = V[f(\cdot, t)] = \max_{\{u(\cdot)\} \in I} \int_t^{t+\Delta} e^{-r(s-t)} U(c(s)) ds + e^{-r\Delta} \left[ P(\tau^1 \leq \Delta) \left[ \Gamma(V(t + \Delta)) + V(t + \Delta) \Phi(V(t + \Delta)) \right] + P(\tau^1 > \Delta) V(t + \Delta) \right]
\]

(5.5)

subject to the law of motion of debt (2.1), the budget constraint (2.2) and the bond pricing equation (5.4). Problem (5.5) describes a deterministic commitment problem over an arbitrary finite-length interval \((t, t + \Delta]\). Default may only happen at the end of the interval provided that an option to default shock has arrived (with probability \(P(\tau^1 \leq \Delta)\)) and that the value after default is higher than the continuation value in the case of no default at the beginning of the next interval \((t + \Delta, t + 2\Delta) : V^D > V(t + \Delta)\). In the case of a default, the expected terminal value is \(\Gamma(V(t + \Delta))\). Notice that \(V(t) : = V[f(\cdot, t)]\) is the aggregate value functional of the problem, which depends on time through its dependence on the distribution.
Main Result. The solution to the government problem (5.2) is the symmetric Markov Perfect Stackelberg Equilibrium that is defined as the limit as \( \Delta \to 0 \) of the family of problems in (5.5). The necessary conditions of a solution to (5.2) are given by the following proposition.

**Proposition 5.1.** If a solution to problem 5.2 exists with \( e^{-\rho t} f, e^{-\rho t} c \in L^2 ([0,T] \times [0,\infty)) \) and \( e^{-\rho t} \in L^2([0,\infty)) \), it should satisfy the PDE

\[
\begin{align*}
  r(t) v(\tau, t) &= -\delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau} - \theta [1 - \Phi(V(t))] v(\tau, t), \text{ if } \tau \in (0, T], \\
  v(0, t) &= -1, \text{ if } \tau = 0,
\end{align*}
\]

with the transversality condition \( \lim_{t \to \infty} e^{-\rho t} v(\tau, t) = 0 \), where \( v(\tau, t) \) is the marginal value of a unit of debt with time-to-maturity \( \tau \), \( e^{-\rho t} v \in L^2 ([0,T] \times [0,\infty)) \) and the optimal issuance \( \iota(\tau, t) \) is given by the optimality condition

\[
\frac{\partial q}{\partial t}(\tau, t) + q(t, \tau, \iota, f) = -v(\tau, t).
\]

The value of \( V(t) \) is the solution of the HJB equation

\[
\begin{align*}
  \rho V[f(\cdot, t)] &= U(c(t)) + \int_0^T U'(c(t)) v(\tau, t) \frac{\partial f}{\partial t} d\tau \\
  &\quad - \theta \{ [1 - \Phi(V[f(\cdot, t)])] V[f(\cdot, t)] - \Gamma(V[f(\cdot, t)]) \}
\end{align*}
\]

and the price of the bond

\[
\begin{align*}
  \tilde{r}\psi(\tau, t) &= \delta + \frac{\partial \psi}{\partial t} - \frac{\partial \psi}{\partial \tau} - \theta [1 - \Phi(V(t))] \psi(\tau, t), \text{ if } \tau \in (0, T) \\
  \psi(0, t) &= 1, \text{ if } \tau = 0.
\end{align*}
\]

**Proof.** See Appendix.

There are four points worth noting regarding the necessary conditions for optimal debt and consumption policies. First, the introduction of the option to default on debt affects the valuation of debt for the investor but also for the government. For international investors, the fact that the possibility of default implies a lower value of debt is common to any model of credit risk. This is well understood. However, for the government, the cost of issuing a bond of maturity \( \tau \) and time \( t \) is now lower, because now the discount factor is \( r(t) + \theta [1 - \Phi(V(t))] \). This is intuitive: the government takes into account that a unit of debt will be defaulted under some states of nature, and will as a consequence,
imply lower expenses in terms of resources honoring this debt. This is a unique feature of our model since we can compare the cost in terms of resources of the different possibilities for the government. Second, note that even though the two valuations decrease unambiguously, this has an ambiguous effect over issuances. For example, in the quadratic case, issuances are given by \[ \iota(\tau, t) = \left( v(\tau, t) + \psi(\tau, t) \right) / \bar{\lambda}, \]

it is clear that the total effect will depend on the relative changes.\(^\text{12}\) Third, the value functional for the government in equilibrium is now modified. To the terms in the perfect foresight model, we need to add one that takes into account the fact that the government is defaulting under some states of nature. Fourth, and finally, note that the stochastic option to default again introduces a computational challenge to compute exact transitional dynamics. This because we need to compute a fixed point in a sequence of distributions and it is numerically unfeasible without an approximation. The government wants to find the optimal threshold to default at each point in time, but the threshold at \( t \) depends on the threshold at \( t' \), and so on and so forth. In order to overcome this problem, we will analyze the particular case in which the economy is in the risky steady state before the arrival of the option to default. As in Section 4, we study this case because we can obtain an exact solution for the transitional dynamics.

5.2 The risky steady state

In this case, the value functional \( V[f(\cdot)] \) is a constant \( \bar{V} \) satisfying:

\[
[\rho + \theta \left( [1 - \Phi(\bar{V})][\bar{V} - \Gamma(\bar{V})] \right)] \bar{V} = \mathcal{U}(c),
\]

and the individual value and bond prices are

\[
v(\tau) = -\delta \left( 1 - e^{-\{\rho + \theta[1 - \Phi(\bar{V})]\}\tau} \right) - e^{-\{\rho + \theta[1 - \Phi(\bar{V})]\}\tau} \frac{1 - e^{-\{\rho + \theta[1 - \Phi(\bar{V})]\}\tau}}{\rho + \theta[1 - \Phi(\bar{V})]},
\]

\[
\psi(\tau) = \frac{\delta \left( 1 - e^{-\{\rho + \theta[1 - \Phi(\bar{V})]\}\tau} \right)}{\bar{\tau} + \theta[1 - \Phi(\bar{V})]} + e^{-\{\rho + \theta[1 - \Phi(\bar{V})]\}\tau}. \]

For the quadratic case, issuances are again given by

\[
\iota(\tau) = \frac{\psi(\tau) + v(\tau)}{\bar{\lambda}}.
\]

\(^{12}\)In the risky steady state that we will analyze below, default will have an un-ambiguous effect.
The first result that we can show is that, as opposed to the case when there are no default opportunities, the maturity of debt shortens. Denote by \( l^\theta=0(\tau) \) the optimal debt issuance’s for a model when there are no default opportunities. Denote by \( l^\theta>0(\tau) \) the optimal debt issuance’s for that same model with a positive probability of a default opportunity. Then we can show the following.

**Corollary 5.1.** When issuance costs are quadratic, issuance’s are given by

\[
l^\theta>0(\tau) \approx e^{-\theta[1-\Phi(\bar{V})]}l^\theta=0(\tau)
\]

with equality when \( \delta = 0 \).

### 5.3 Numerical results

Consider the same calibration as in the deterministic case and assume that \( \theta = 0.02 \) and \( V^D \) is uniformly distributed between \([-b,-a]\), with \( a, b > 0 \). We calibrate the distribution such that \( \Phi(\bar{V}) = \bar{V}+b \). In this case \( \Gamma(\bar{V}) := \frac{1}{a-b} \int_{-a}^{-\bar{V}} d\xi = \frac{1}{2} \left( \frac{a^2-b^2}{a-b} \right) \).

### 5.4 The case with a single default opportunity

In order to have an easier comparison with the model in Section 4, where the shocks to output and the interest rate can hit only one time, we now consider instead the case in which the opportunity to default only arrives once. In this case, if the country does no default when the opportunity arrives, it just follows a deterministic path from then on, characterized by the solution of the model with Perfect Foresight. Let’s define \( \hat{V}(t) := \hat{V}[f(\cdot,t)] \) as the aggregate value functional in the deterministic case if the debt distribution is \( f(\cdot,t) \). The expected value if the shock arrives between \( t \) and \( t+\Delta \) is given by

\[
\Gamma(\hat{V}(t+\Delta)) + \hat{V}(t+\Delta) \Phi(\hat{V}(t+\Delta)).
\]

The necessary conditions for an optimal plan are given by the following proposition.

**Proposition 5.2.** If a solution to problem with one-off exists with \( e^{-\rho t}f, e^{-\rho t}l \in L^2([0,T] \times [0,\infty)) \) and \( e^{-\rho t}c \in L^2[0,\infty) \) it should satisfy the PDE

\[
\begin{align*}
r(t)v(\tau,t) &= -\delta + \frac{\partial v}{\partial t} - \theta \left[ v(\tau,t) - \Phi(\hat{V}[f(\cdot,t)]) \tilde{v}(\tau,t) \frac{U'(\tilde{c}(t))}{U'(c(t))} \right], \text{ if } \tau \in (0,T], \\
v(0,t) &= -1, \text{ if } \tau = 0,
\end{align*}
\]

with the transversality condition \( \lim_{t \to \infty} e^{-\rho t}v(\tau,t) = 0 \), where \( v(\tau,t) \) is the marginal value of a unit of debt with time-to-maturity \( \tau \), \( e^{-\rho t}v \in L^2([0,T] \times [0,\infty)) \) and \( \tilde{c}(t) \) and \( \tilde{v}(\tau,t) \) are
respectively consumption and the marginal value of a unit of debt with time-to-maturity $\tau$ in the no-default case. The optimal issuance $i(\tau, t)$ is given by the optimality condition (5.6). The value of $\tilde{V}(t)$ is the solution of the HJB equation in the no-default case

$$\rho \tilde{V}(t) = U(\tilde{c}(t)) + \int_0^T U'(\tilde{c}(t)) \tilde{v}(\tau, t) \frac{\partial f}{\partial t} d\tau.$$  \hspace{1cm} (5.7)

**Proof.** See Appendix. \hfill \Box

**Remarks.** There are three points worth noting. First, note that with intensity $\theta$ a default option arrives. In this case, the government will default in case it draws a value lower than $\tilde{V}(\cdot, t)$. This occurs with probability $\Phi(\tilde{V}(\cdot, t))$. In this case, the government valuation of the bond is given by $\bar{v}(\tau, t)$, the valuation of a bond given by the solution of the perfect foresight model. Second, note in addition, that there is a correction term that accounts for the change in the marginal utilities. This term is given by $U'(\tilde{c}(t))/U'(c(t))$, because consumption will in general jump after the shock has hit. Third, note that the bond prices in this case are given by

$$\bar{r}\psi(\tau, t) = \delta + \frac{\partial \psi}{\partial t} - \frac{\partial \psi}{\partial \tau} - \theta \left[ \psi(\tau, t) - \Phi(\tilde{V}(t)) \tilde{v}(\tau, t) \right], \text{ if } \tau \in (0, T)$$

$$\psi(0, t) = 1, \text{ if } \tau = 0,$$

where

$$\tilde{v}(\tau) = \frac{\delta (1 - e^{-\bar{r}\tau})}{\bar{r}} + e^{-\bar{r}\tau},$$

is the riskless bond price with a flat yield curve. The prices take into account that with arrival $\theta$, a default opportunity arises, and the government takes it only if the value of continuation is lower than the realized outside options. This occurs with probability $\Phi(\tilde{V}(t))$, where $\tilde{V}(t)$ is the value of repayment if no more shock can arrive; this value if characterized by Section 3, Perfect foresight.

**Risky Steady State.** The risky steady state valuations are given by

$$(\rho + \theta) v(\tau) = -\delta - \frac{\partial v}{\partial \tau} + \theta \left[ \Phi(\tilde{V}(\cdot)) \tilde{v}(\tau, 0) \frac{U'(\tilde{c}(0))}{U'(c)} \right], \tau \in (0, T),$$

$$v(0) = -1, \text{ if } \tau = 0,$$
and a bond price

\[(\bar{r} + \theta)\psi(\tau) = \delta - \frac{\partial \psi}{\partial \tau} + \theta \left[ \Phi (\tilde{V} (t)) \dot{\psi}(\tau) \right], \text{ if } \tau \in (0, T)\]

\[\psi(0, t) = 1, \text{ if } \tau = 0.\]

and an aggregate value functional

\[\tilde{V} [f (\cdot)] = \frac{1}{\rho} \left\{ U (\tilde{c} (0)) + U' (\tilde{c} (0)) \int_{0}^{T} \tilde{\varphi} (\tau, 0) \frac{\partial f}{\partial \tau} d\tau \right\}.\]

As opposed to the case in which there is a recurrent default arrival, the difference is in the continuation payoffs, both for the government and the international investors, if the government does not default. In this case, they are given by the values of the bonds, for both agents, with an initial distribution and if there are no more default opportunities arriving; the Perfect foresight case.
References


Nuño, Galo and Benjamin Moll, “Controlling a Distribution of Heterogeneous Agents,” 2015.


Figure A.1: The figure describes the steady state values of debt, issuances, maturity distribution and the wedges in valuations.
Figure A.2: The figure describes the response of the government discount factor, issuances, consumption, and total debt, from an unexpected shock to output of 5% that reverts to the long run mean as an AR(1) deterministic process.
Figure A.3: The figure describes the response of the government discount factor, issuances, consumption, and total debt, from an unexpected shock to output of 5% that reverts to the long run mean as an AR(1) deterministic process, when $T$ increases from 20 to 30 years.
Figure A.4: The figure describes the response of the yield curve to a shock in the short rate that reverts as an AR(1) deterministic process.
Figure A.5: The figure describes the response of the government discount factor, issuances, consumption, and total debt, from an unexpected shock to the short interest rate that reverts to the long run mean as an AR(1) deterministic process.
Figure A.6: The figure describes the response of the government discount factor, issuances, consumption, and total debt, from an unexpected shock to the short interest rate that reverts to the long run mean as an AR(1) deterministic process, for the case in which the households are risk neutral.
Figure A.7: Response, in the risky steady state, of the maturity distribution, total debt, issuances, and valuations to an expected 10 percent drop in output that reverts back to normal in the long run as an AR(1) deterministic process.
Figure A.8: Dynamics total debt, issuances, consumption and the internal rate after a 10 percent shock in output that reverts to the long run mean as an AR(1) deterministic process.
Figure A.9: Response, in the risky steady state, of the maturity distribution, total debt, issuances, and valuations to an expected permanent increase in the short rate.
Figure A.10: Dynamics total debt, issuances, consumption and the internal rate to an expected permanent increase in the short rate.
B Proofs

B.1 Proof of Proposition 3.1

First we construct a Lagrangian in the space of functions $g$ such that $\|e^{-\rho r/2}g(\tau, t)\|_{L^2} < \infty$. The Lagrangian is

$$
\mathcal{L}(\iota, f) = \int_0^\infty e^{-\rho \iota t} \left( y(t) - f(0, t) + \int_0^T [q(\tau, t, \iota, f) \iota(\tau, t) - \delta f(\tau, t)] d\tau \right) dt
$$

$$
+ \int_0^\infty \int_0^T e^{-\rho \iota t} j(\tau, t) \left( -\frac{\partial f}{\partial t} + i(\tau, t) + \frac{\partial f}{\partial \tau} \right) d\tau dt,
$$

where $j(\tau, t)$ is the Lagrange multiplier associated to the law of motion of debt. Taking Gateaux derivatives, for any suitable $h(\tau, t)$ such that $e^{-\rho \iota t}h \in L^2([0, T] \times [0, \infty))$:

$$
\lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} \mathcal{L}(\iota, f + \alpha h) = \int_0^\infty e^{-\rho \iota t} U'(c) \left[ -h(0, t) + \int_0^T \left( \frac{\partial q}{\partial \iota} i(\tau, t) - \delta \right) h(\tau, t) d\tau \right] dt
$$

$$
- \int_0^\infty \int_0^T e^{-\rho \iota t} \frac{\partial h}{\partial t} j(\tau, t) d\tau dt
$$

$$
+ \int_0^\infty \int_0^T e^{-\rho \iota t} \frac{\partial h}{\partial \iota} j(\tau, t) d\tau dt,
$$

The last two terms can be integrated by parts

$$
\int_0^T \frac{\partial h}{\partial \iota} j(\tau, t) d\tau = h(T, t) j(T, t) - h(0, t) j(0, t) - \int_0^T h(\tau, t) \frac{\partial j}{\partial \tau} d\tau,
$$

$$
- \int_0^\infty e^{-\rho \iota t} \frac{\partial h}{\partial t} j(\tau, t) dt = - \lim_{s \to \infty} e^{-\rho s} h(\tau, s) j(\tau, s) + h(\tau, 0) j(\tau, 0) + \int_0^\infty e^{-\rho \iota t} h(\tau, t) \left( \frac{\partial j}{\partial \tau} - \rho j \right) d\tau.
$$

As the initial distribution $f_0$ is given the value of $h(\tau, 0) = 0$. The Gateaux derivative should be zero for any suitable $h(\tau, t)$

$$
0 = \int_0^\infty e^{-\rho \iota t} U'(c) \left[ -h(0, t) + \int_0^\infty \left( \frac{\partial q}{\partial \iota} i(\tau, t) - \delta \right) h(\tau, t) d\tau \right] d\tau
$$

$$
+ \int_0^\infty \int_0^T e^{-\rho \iota t} \left( -\rho j - \frac{\partial j}{\partial \iota} + \frac{\partial j}{\partial t} \right) h(\tau, t) d\tau dt
$$

$$
\int_0^\infty e^{-\rho \iota t} (h(T, t) j(T, t) - h(0, t) j(0, t)) dt
$$

$$
- \int_0^\infty \lim_{s \to \infty} e^{-\rho s} h(\tau, s) j(\tau, s) d\tau.
$$
Therefore, as $f(T^+, t) = 0$ then $h(T, t) = 0$ and we have

$$
\rho j(\tau, t) = U'(c(t)) \left( \frac{\partial q}{\partial f} \right) t - \delta + \frac{\partial j}{\partial t} - \frac{\partial j}{\partial \tau}, \text{ if } \tau \in (0, T)
$$

(B.1)

$$
j(0, t) = -U'(c(t)), \text{ if } \tau = 0,
$$

$$
\lim_{t \to \infty} e^{-\rho t} j(\tau, t) = 0.
$$

Proceeding similarly in the case of $\iota$

$$
\lim_{a \to 0} \frac{\partial}{\partial \alpha} \mathcal{L}(\iota + \alpha h, f) = \int_0^\infty e^{-\rho t} U'(c) \left[ \int_0^T \left( \frac{\partial q}{\partial \iota} \tau(t, \tau) + q(t, \tau, \iota, f) \right) h(\tau, t) d\tau \right] d\tau
$$

$$
+ \int_0^\infty \int_0^T e^{-\rho t} h(\tau, t) j(\tau, t) d\tau dt,
$$

and hence

$$
U'(c) \left( \frac{\partial q}{\partial \iota} \tau(t, \tau) + q(t, \tau, \iota, f) \right) = -j(\tau, t).
$$

If we define the variable

$$
v(\tau, t) = j(\tau, t) / U'(c(t)),
$$

the PDE equation (C.1) results in

$$
\left( \rho - \frac{U''(c(t))}{U'(c(t))} \right) \frac{dc}{dt} v(\tau, t) = \frac{\partial q}{\partial \iota} \tau - \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \text{ if } \tau \in (0, \infty),
$$

$$
v(0, t) = -1, \text{ if } \tau = 0,
$$

$$
\lim_{t \to \infty} e^{-\rho t} v(\tau, t) = 0,
$$

and the first order condition is

$$
\frac{\partial q}{\partial \iota} \tau(t, \tau) + q(t, \tau, \iota, f) = -v(\tau, t).
$$
B.2 Proof of Proposition 4.1

To compute the dynamics of $\iota [f (\cdot, t), y^H]$ we need to employ dynamic programming. The value functional can be expressed as

$$V[f (\cdot, t), y^H] = \max_{\{\iota (\cdot)\} \in \mathcal{I}} \mathbb{E}_t \left[ \int_t^\infty e^{-\rho(s-t)} \left[ e^{-\phi(s-t)} \mathcal{U}(c(s)) + \left( 1 - e^{-\phi(s-t)} \right) V[f (\cdot, s), y^L] \right] ds \mid y_t = y^H \right]$$

We first apply Bellman’s Principle of Optimality

$$V[f (\cdot, t), y^H] = \max_{\{\iota (\cdot)\} \in \mathcal{I}} \mathbb{E}_t \left[ \int_t^{t'} e^{-\rho(s-t)} \left[ e^{-\phi(s-t)} \mathcal{U}(c(s)) + \left( 1 - e^{-\phi(s-t)} \right) V[f (\cdot, s), y^L] \right] ds \mid y_t = y^H \right]$$

+ $e^{-\rho(t'-t)} \mathbb{E}_t \left[ e^{-\phi(t'-t)} V[f (\cdot, t'), y^H] + \left( 1 - e^{-\phi(t'-t)} \right) V[f (\cdot, t'), y^L] \mid y_t = y^H \right]$, for an arbitrary $t < t'$ and then we take the derivative with respect to $t'$ and the limit $t' \to t$:

$$\rho V[f (\cdot, t)] = \max_{\{\iota (\cdot)\} \in \mathcal{I}} \mathcal{U}(c(t)) + \frac{1}{dt} dV[f (\cdot, t)] + \phi \left[ \hat{V}[f (\cdot, t)] - V[f (\cdot, t)] \right], \quad (B.2)$$

where $\hat{V}[f (\cdot, t)] := V[f (\cdot, t), y^L]$ and $V[f (\cdot, t)] := V[f (\cdot, t), y^H]$. We work with the HJB (B.2) in the space of functions $L^2 \left( [0, T] \times [0, \infty) \right)$. If we apply the Riesz representation theorem the Gateaux derivative of $V$ with respect to $f$ in the direction $h (\tau) \in L^2 \left( [0, T] \right)$ can be expressed as

$$\frac{\partial}{\partial \alpha} V[f (\cdot, t) + \alpha h|_{\alpha = 0} = \int_0^T \delta V \frac{\partial f (\cdot, t)}{\partial \tau} h (\tau) d\tau,$$

where $\delta V \in L^2 \left( [0, T] \right)$. Therefore

$$\frac{1}{dt} dV[f (\cdot, t)] = \int_0^T \delta V \frac{\partial f (\cdot, t)}{\partial \tau} \frac{\partial f (\cdot, t)}{\partial \tau} d\tau,$$
where we have applied the chain rule. The HJB (B.2) can thus be expressed as

\[
(\rho + \phi) V[f(\cdot, t)] = \max_{\nu(\cdot, t)} U(c(t)) + \int_{0}^{T} \delta V \frac{\partial f}{\partial t} d\tau \tag{B.3}
\]

\[
= \max_{\nu(\cdot, t)} \left( y(t) - f(0, t) + \int_{0}^{T} [q(\tau, t, \nu) \nu(t, t) - \delta f(\tau, t)] d\tau \right) + \int_{0}^{T} \frac{\delta V}{\delta f} \left( \nu(t, t) + \frac{\partial f}{\partial \tau} \right) d\tau + \phi \hat{V}[f(\cdot, t)].
\]

where we have substituted the budget constraint and the KFE.

The first order condition with respect to \(\nu\) can be obtained by computing the Gateaux derivative in (B.3):

\[
0 = \frac{\partial}{\partial \alpha} \left. U \left( y(t) - f(0, t) + \int_{0}^{T} [q(\tau, t, \nu + \alpha h) (\nu + \alpha h) - \delta f(\tau, t)] d\tau \right) \right|_{\alpha=0} + \frac{\partial}{\partial \alpha} \left. \int_{0}^{T} \frac{\delta V}{\delta f} \left( \nu + \alpha h + \frac{\partial f}{\partial \tau} \right) d\tau \right|_{\alpha=0},
\]

and

\[
0 = U'(c) \left[ \int_{0}^{T} \left( \frac{\partial q}{\partial t} \nu(t, t) + q(\tau, t, \nu) \right) h(\tau) d\tau \right] dt + \int_{0}^{T} \frac{\delta V}{\delta f} h(\tau) d\tau.
\]

The Gateaux derivative should be zero for any suitable \(h(\tau) \in L^2[0, T]\)

\[
U'(c) \left( \frac{\partial q}{\partial t} \nu(t, t) + q(\tau, t, \nu) \right) = -\frac{\delta V}{\delta f}.
\]

If we define

\[
j(\tau, t) := \frac{\delta V}{\delta f}, \tag{B.4}
\]

---

\[^{13}\text{Notice that}
\]

\[
\left. \frac{1}{dt} dV[f(\cdot, t)] \right| = \lim_{a \to 0} \frac{V[f(\cdot, t + a)] - V[f(\cdot, t)]}{a} = \lim_{a \to 0} \frac{V[f(\cdot, t) + a \partial f(\cdot, t) / \partial t] - V[f(\cdot, t)]}{a} = \frac{\partial}{\partial \alpha} \left. \frac{V[f(\cdot, t) + a \partial f(\cdot, t) / \partial t]}{a} \right|_{\alpha=0}.
\]
then the first order condition results in
\[ \mathcal{U}'(c) \left( \frac{\partial q}{\partial t}(\tau, t) + q(\tau, t, \iota) \right) = -j(\tau, t). \]

If we compute Gateaux derivatives with respect to \( f \) in the HJB equation (B.3) we obtain
\[
0 = \frac{\partial}{\partial \alpha} \mathcal{U} \left( y(t) - (f(0,t) + ah(0)) + \int_0^T \left[ q + -\delta(f + ah) \right] d\tau \right) \bigg|_{\alpha=0} \\
+ \frac{\partial}{\partial \alpha} \int_0^T \frac{\delta V}{\delta (f + ah)} \left( I + \frac{\partial (f + ah)}{\partial \tau} \right) d\tau \bigg|_{\alpha=0} \\
- (\rho + \phi) \frac{\partial}{\partial \alpha} V [f(\cdot,t) + ah] \bigg|_{\alpha=0} + \phi \frac{\partial}{\partial \alpha} \hat{V} [f(\cdot,t) + ah] \bigg|_{\alpha=0}
\]
and
\[
0 = \mathcal{U}'(c) \left[ -h(0,t) + \int_0^T (-\delta) h(\tau) d\tau \right] \\
+ \int_0^T \int_0^T \frac{\delta^2 V}{\delta f^2} \left( I + \frac{\partial f}{\partial \tau} \right) h(\tau') d\tau d\tau' \\
+ \int_0^T \frac{\delta V}{\delta f} \frac{\partial h}{\partial \tau} d\tau - (\rho + \phi) \int_0^T \frac{\delta V}{\delta f} h(\tau) d\tau \\
+ \phi \int_0^T \frac{\delta \hat{V}}{\delta f} h(\tau) d\tau,
\]
where \( \hat{j}(\tau, t) := \frac{\delta \hat{V}}{\delta f} \) corresponds to the value function in the case of \( y_t = y^L \). The term \( \int_0^T \frac{\delta^2 V}{\delta f^2} h(\tau') d\tau' \) is the second Gateaux derivative of \( V \). There is a mapping between \( \frac{\delta^2 V}{\delta f^2} \) and \( \frac{\partial j}{\partial \tau} \), where \( j \) has been defined in (B.4):
\[
\frac{\partial j}{\partial t} = \frac{\partial}{\partial t} \frac{\delta V}{\delta f} = \int_0^T \frac{\delta^2 V}{\delta f^2} \frac{\partial f}{\partial t} d\tau = \int_0^T \frac{\delta^2 V}{\delta f^2} \left( I + \frac{\partial f}{\partial \tau} \right) d\tau,
\]
where we have applied again the chain rule and the Riesz representation theorem. The
first order condition with respect to \( f \) then results in
\[
0 = \mathcal{U}'(c) \left[ -h(0,t) + \int_0^T (-\delta) h(\tau) d\tau \right] \\
+ \int_0^T \left[ \frac{\partial j}{\partial t} - (\rho + \phi) j(\tau, t) + \phi \hat{j}(\tau, t) \right] h(\tau) d\tau \\
+ \int_0^T j(\tau, t) \frac{\partial h}{\partial \tau} d\tau
\]
Integrating by parts we obtain
\[
0 = U'(c) \left[ -h(0,t) + \int_0^T (-\delta) h(\tau) \, d\tau \right] 
+ \int_0^T \left( \frac{\partial j}{\partial t} - \frac{\partial j}{\partial \tau} - (\rho + \phi) j(\tau,t) + \phi \hat{j}(\tau,t) \right) h(\tau) \, d\tau 
+ j(T,t) h(T) - j(0,t) h(0).
\]

As \( f(T^+,t) = 0 \) then \( h(T) = 0 \). As the Gateaux derivative should be zero for any suitable \( h(\tau) \):
\[
(\rho + \phi) j(\tau,t) = U'(c(t)) (-\delta) + \frac{\partial j}{\partial t} - \frac{\partial j}{\partial \tau} + \phi \hat{j}(\tau,t), \text{ if } \tau \in (0,T) \quad (B.5)
\]
\[
        j(0,t) = -U'(c(t)), \text{ if } \tau = 0.
\]

The transversality condition derives from the fact that \( \lim_{t \to \infty} e^{-\rho t} V[f(\cdot,t)] = 0 \) and thus, if we take Gateaux derivatives at both sides:
\[
\lim_{t \to \infty} e^{-\rho t} j(\tau,t) = 0.
\]

Finally, if we define the variables
\[
\nu(\tau,t) = j(\tau,t) / U'(c^H(t)), \\
\hat{\nu}(\tau,t) = \hat{j}(\tau,t) / U'(c^L(t)),
\]
where \( c^H(t) \) and \( c^L(t) \) correspond to the optimal consumption in the case of \( y(t) = y^H \) or \( y(t) = y^L \) respectively, the PDE equation (B.5) results in
\[
\left( \rho - \frac{U''(c^H(t))}{U'(c^H(t))} \frac{dc^H}{dt} \right) \nu(\tau,t) = -\delta + \frac{\partial \nu}{\partial t} - \frac{\partial \nu}{\partial \tau} + \phi \hat{\nu}(\tau,t) U'(c^L(t)) / U'(c^H(t)), \text{ if } \tau \in (0,\infty),
\]
\[
\nu(0,t) = -1, \text{ if } \tau = 0,
\]
\[
\lim_{t \to \infty} e^{-\rho t} \nu(\tau,t) = 0,
\]
and the first order condition is
\[
\frac{\partial q}{\partial t} l(\tau,t) + q(\tau,t,i) = -\nu(\tau,t).
\]
A. Proof of Proposition 5.1

The solution strategy proceeds in three steps. We start by solving the problem with commitment in an arbitrary interval \((t, t + \Delta]\), then we take the limit as \(\Delta \to 0\), and finally we employ dynamic programming to obtain the value of the value functional \(V[f(\cdot, t)]\).

Step 1. The problem with commitment

Assume first that the time interval \([0, \infty)\) is divided in subintervals \([0, \Delta] \cup (\Delta, 2\Delta] \cup (2\Delta, 3\Delta] \cup \ldots\) such that the shock may potentially arrive at the beginning of each interval (except at time \(t = 0\)) and that the government solves a commitment problem in each subinterval. The solution procedure to problem (??) follows the same lines as in the perfect foresight case. The Lagrangian is given by:

\[
L[\nu, f, \psi] = \int_t^{t+\Delta} e^{-\rho(s-t)} U(y - f(0,s) + \int_0^T \{[\psi(\tau,s) + \lambda(\tau,s,t)]t(\tau,s) - \delta f(\tau,s)\} \, d\tau) \, ds
+ e^{-\rho\Delta} \left(1 - e^{-\theta\Delta}\right) [\Gamma(V[f(\cdot,s+\Delta)]) + V[f(\cdot,t+\Delta)] \Phi(V[f(\cdot,t+\Delta)])
+ e^{-\rho\Delta} e^{-\theta\Delta} V[f(\cdot,t+\Delta)]
+ \int_t^{t+\Delta} \int_0^T e^{-\rho(s-t)} j(\tau,s) \left(-\frac{\partial f}{\partial s} + t(\tau,s) + \frac{\partial f}{\partial \tau}\right) \, d\tau ds
+ \int_t^{t+\Delta} \int_0^T e^{-\rho(t-s)} \mu(\tau,s) \left(-\beta \psi(\tau,s) + \delta + \frac{\partial \psi}{\partial s} - \frac{\partial \psi}{\partial \tau}\right) \, d\tau ds
\]

where \(j(\tau,t)\) and \(\mu(\tau,t)\) are the Lagrange multipliers associated to the law of motion of debt (2.1) and bond pricing equation (??), respectively. We integrate by parts the terms
that involve derivatives of $f$ and $\psi$:

$$
- \int_0^{t+\Delta} \int_0^T e^{-\rho(s-t)} f(t, s) \frac{\partial f}{\partial s} j(\tau, s) ds \, d\tau = - \int_0^T e^{-\rho \Delta} f(\tau, t + \Delta) j(\tau, t + \Delta) \, d\tau
+ \int_0^T f(\tau, t) j(\tau, t) \, d\tau
+ \int_t^{t+\Delta} \int_0^T e^{-\rho(s-t)} f \left( \frac{\partial j}{\partial s} - \rho j \right) ds \, d\tau
$$

$$
\int_t^{t+\Delta} \int_0^T \frac{\partial f}{\partial \tau} j(\tau, s) d\tau ds = \int_t^{t+\Delta} e^{-\rho(s-t)} f(T, s) j(T, s) ds
- \int_t^{t+\Delta} e^{-\rho(s-t)} f(0, s) j(0, s) ds
- \int_t^{t+\Delta} \int_0^T e^{-\rho(t-s)} f \frac{\partial j}{\partial s} ds \, d\tau
$$

$$
\int_t^{t+\Delta} \int_0^T e^{-\rho(t-s)} \mu(\tau, s) \left( - \frac{\partial \psi}{\partial \tau} \right) d\tau ds = \int_t^{t+\Delta} e^{-\rho(s-t)} \mu(T, s) \psi(T, s) - e^{-\rho t} \mu(0, s) \psi(0, s)
+ \int_t^{t+\Delta} \int_0^T e^{-\rho t} \psi(\tau, s) \frac{\partial \mu}{\partial s} \, d\tau ds,
$$

$$
\int_t^{t+\Delta} \int_0^T e^{-\rho(t-s)} \mu(\tau, s) \frac{\partial \psi}{\partial s} \, d\tau ds = \int_0^T \left[ e^{-\rho \Delta} \mu(\tau, t + \Delta) \psi(\tau, t + \Delta) - \mu(\tau, t) \psi(\tau, t) \right] d\tau
- \int_t^{t+\Delta} \int_0^T e^{-\rho t} \psi(\tau, s) \left( \frac{\partial \mu}{\partial s} - \rho \mu \right) \, d\tau ds
$$

and substitute the terminal condition for the prices and distribution:

$$
\psi(\tau, t + \Delta) = \left[ \left( 1 - e^{-\theta \Delta} \right) \Phi \left( V(f(\cdot, t + \Delta)) \right) + e^{-\theta \Delta} \right] \psi(\tau, f(\cdot, t + \Delta)), \quad (B.6)
$$

$$
\psi(0, s) = 1,
$$

$$
f(T, s) = 0.
$$
Going back to the Lagrangian, taking Gateaux derivative with respect to \( h (\tau, t) \) such that \( e^{-\rho t} h \in L^2 ([0, T] \times (t, t + \Delta]) : \frac{\partial}{\partial \alpha} \mathcal{L} [f + \alpha h] |_{\alpha=0} \) equals

\[
\int_t^{t+\Delta} e^{-\rho(s-t)} U'(c) \left[ -h(0,s) + \int_0^T (-\delta h(\tau,s) \, d\tau \right) ds \\
+ e^{-\rho \Delta} \left( 1 - e^{-\theta \Delta} \right) \frac{\partial}{\partial \alpha} \Gamma \left( V [f (\cdot, t + \Delta) + \alpha h (\tau, t + \Delta)] \right) |_{\alpha=0} \\
+ e^{-\rho \Delta} \left( 1 - e^{-\theta \Delta} \right) \frac{\partial}{\partial \alpha} V [f + \alpha h] \Phi (V [f + \alpha h]) |_{\alpha=0} \\
+ e^{-\rho \Delta} e^{-\theta \Delta} \frac{\partial}{\partial \alpha} V [f (\cdot, t + \Delta) f (\cdot, t + \Delta) + \alpha h (\tau, t + \Delta)] |_{\alpha=0} \\
- \int_0^T e^{-\rho \Delta} h (\tau, t + \Delta) j (\tau, t + \Delta) \, d\tau + \int_0^T h (\tau, t) j (\tau, t) \, d\tau \\
+ \int_t^{t+\Delta} \int_0^T e^{-\rho(s-t)} h (\frac{\partial j}{\partial s} - \rho j) \, ds \, d\tau \\
+ \int_t^{t+\Delta} e^{-\rho(s-t)} [h (T, s) j (T, s) - h(0,s) j(0,s)] \, ds - \int_t^{t+\Delta} \int_0^T e^{-\rho(t-s)} h \frac{\partial j}{\partial t} \, d\tau \, ds, \\
+ \int_0^T e^{-\rho \Delta} \left( 1 - e^{-\theta \Delta} \right) \mu (\tau, t + \Delta) \Phi' (V [f (\cdot, t + \Delta)]) \frac{\partial}{\partial \alpha} V [f (\cdot, t + \Delta) + \alpha h (\cdot, t + \Delta)] |_{\alpha=0} \psi (\tau, t + \Delta) \, d\tau \\
+ \int_0^T e^{-\rho \Delta} \left[ 1 - e^{-\theta \Delta} \right] \Phi (V [f (\cdot, t + \Delta)]) e^{-\theta \Delta} \mu (\tau, t + \Delta) \frac{\partial}{\partial \alpha} \psi [\tau, f (\cdot, t + \Delta) + \alpha h (\cdot, t + \Delta)] |_{\alpha=0} \, d\tau,
\]

We can express

\[
\frac{\partial}{\partial \alpha} V [f (\cdot, t + \Delta) + h (\cdot, t + \Delta)] |_{\alpha=0} = \int_0^T \frac{\delta V}{\delta f} (\tau, \tau', t + \Delta) h (\tau', t + \Delta) \, d\tau', \\
\frac{\partial}{\partial \alpha} \tau, \psi [f (\cdot, t + \Delta) + \alpha h (\cdot, t + \Delta)] |_{\alpha=0} = \int_0^T \frac{\delta \psi}{\delta f} (\tau, \tau', t + \Delta) h (\tau', t + \Delta) \, d\tau'.
\]
As the initial distribution \( f (\tau, t) \) is given, the value of \( h (\tau, t) = 0 \). The Gateaux derivative should be zero for any suitable \( h (\tau, t) \)

\[
0 = \int_t^{t+\Delta} e^{-\rho(s-t)} U' (c) \left[ -h (0, s) + \int_0^T (-\delta) h (\tau, s) \, d\tau \right] \, ds \\
- e^{-\rho \Delta} \left( 1 - e^{-\theta \Delta} \right) V [f (\cdot, t + \Delta)] \phi (V [f (\cdot, t + \Delta)]) \int_0^T \frac{\delta V}{\delta f} (\tau, t + \Delta) h (\tau, t + \Delta) \, d\tau \\
+ e^{-\rho \Delta} \left( 1 - e^{-\theta \Delta} \right) V [f (\cdot, t + \Delta)] \phi (V [f (\cdot, t + \Delta)]) \int_0^T \frac{\delta V}{\delta f} (\tau, t + \Delta) h (\tau, t + \Delta) \, d\tau \\
+ e^{-\rho \Delta} \left( 1 - e^{-\theta \Delta} \right) \Phi (V [f (\cdot, t + \Delta)]) \int_0^T \frac{\delta V}{\delta f} (\tau, t + \Delta) h (\tau, t + \Delta) \, d\tau \\
+ e^{-\rho \Delta} e^{-\theta \Delta} \int_0^T \frac{\delta V}{\delta f} (\tau, s) h (\tau, s) \, d\tau \\
- \int_0^T e^{-\rho \Delta} h (\tau, t + \Delta) j (\tau, t + \Delta) + \int_t^{t+\Delta} \int_0^T e^{-\rho(s-t)} h \left( \frac{\partial j}{\partial s} - \rho j \right) \, ds \\
+ \int_t^{t+\Delta} e^{-\rho(s-t)} [h (T, s) j (T, s) - h (0, s) j (0, s)] \, ds - \int_0^T \int_t^{t+\Delta} e^{-\rho(s-t)} h \left( \frac{\partial j}{\partial \tau} \right) \, d\tau \, ds \\
+ e^{-\rho \Delta} \left( 1 - e^{-\theta \Delta} \right) \phi (V [f (\cdot, t + \Delta)]) \int_0^T \mu (\tau, t + \Delta) \psi (\tau, t + \Delta) \int_0^T \frac{\delta V}{\delta f} (\tau', t + \Delta) h (\tau', t + \Delta) \, d\tau' \, d\tau \\
+ e^{-\rho \Delta} \left[ \left( 1 - e^{-\theta \Delta} \right) \Phi (V [f (\cdot, t + \Delta)]) + e^{-\theta \Delta} \right] \int_0^T \mu (\tau, t + \Delta) \int_0^T \frac{\delta \psi}{\delta f} (\tau, \tau', t + \Delta) h (\tau', t + \Delta) \, d\tau' \, d\tau \\
\]

Therefore, as \( f (T^+, t) = 0 \) then \( h (T, t) = 0 \) and we have

\[
\rho j (\tau, s) = U' (c (s)) (-\delta) + \frac{\partial j}{\partial s} - \frac{\partial j}{\partial \tau}, \text{ if } \tau \in (0, T) \\
\]

\[
j (0, s) = -U' (c (s)), \text{ if } \tau = 0, \\
\]

\[
j (\tau, t + \Delta) = \left( 1 - e^{-\theta \Delta} \right) V [f (\cdot, t + \Delta)] \phi (V [f (\cdot, t + \Delta)]) \frac{\delta V}{\delta f} (\tau, t + \Delta) + e^{-\theta \Delta} \frac{\delta V}{\delta f} (\tau, t + \Delta) \\
+ \left( 1 - e^{-\theta \Delta} \right) \phi (V [f (\cdot, t + \Delta)]) \left[ \int_0^T \mu (\tau', t + \Delta) \psi (\tau', t + \Delta) \, d\tau' \right] \frac{\delta V}{\delta f} (\tau, t + \Delta) \\
+ \left( 1 - e^{-\theta \Delta} \right) \left[ \left( 1 - e^{-\theta \Delta} \right) \Phi (V [f (\cdot, t + \Delta)]) + e^{-\theta \Delta} \right] \int_0^T \mu (\tau', t + \Delta) \frac{\delta \psi}{\delta f} (\tau', \tau, t + \Delta) \, d\tau'. \\
\]

Taking Gateaux derivative with respect to \( t \) the results are the same as in the deterministic case:

\[
U' (c) \left( \frac{\partial q}{\partial t} (\tau, t) + q (t, \tau, t, f) \right) = -j (\tau, t). \\
\]
Finally, we need to the Gateaux derivatives with respect to $\psi$:

$$
\frac{\partial}{\partial \alpha} L [\psi + \alpha h] |_{\alpha = 0} = \int_t^{t+\Delta} e^{-\rho(s-t)} U'(c) \left[ \int_0^T \iota(\tau, s) h(\tau, s) d\tau \right] ds
$$

$$
+ \int_t^{t+\Delta} e^{-\rho(s-t)} \mu(T, s) h(T, s) ds - e^{-\rho t} \mu(0, s) h(0, s)
$$

$$
+ \int_t^{t+\Delta} \int_0^T e^{-\rho t} h(\tau, s) \frac{\partial \mu}{\partial \tau} d\tau ds
$$

$$
- \int_0^T \mu(\tau, t) h(\tau, t) d\tau
$$

$$
- \int_t^{t+\Delta} \int_0^T e^{-\rho t} \left( \frac{\partial \mu}{\partial s} - \rho \mu \right) h(\tau, s) d\tau ds,
$$

where $h(0, s) = 0$ as the value of $\psi(0, s)$ is fixed to 1. And the optimality condition then results in

$$
\frac{\partial \mu}{\partial s} = \rho \mu(\tau, s) + U'(c) \iota(\tau, s) + \frac{\partial \mu}{\partial \tau}, \text{ if } \tau \in (0, T), s \in (t, t + \Delta)
$$

$$
\mu(T, s) = 0, \text{ if } \tau = T, s \in (t, t + \Delta)
$$

$$
\mu(\tau, t) = 0, \text{ if } \tau \in (0, T).
$$

(B.9)

**Step 2: the instantaneous limit**

The second step is to take the limit as $\Delta \to 0$. Before that, we express the HJB equation (B.7) as

$$
j(\tau, t) = \int_t^{t+\Delta} e^{-\rho(t-s)} \left\{ U'(c(s)) (-\delta) \right\} ds
$$

$$
+ e^{-\rho \Delta} E_t \left[ \frac{\delta V}{\delta f} (\tau, t + \Delta) + \int_0^T \mu(\tau', t + \Delta) \frac{\delta \psi}{\delta f} (\tau', \tau, t + \Delta) d\tau' \right],
$$

(B.11)

$$
j(0, s) = -U'(c(s)), \text{ if } \tau = 0,
$$

where the term $\frac{\delta V}{\delta f}(\tau, t + \Delta)$ depends on the arrival of a shock with probability $(1 - e^{-\theta \Delta})$ that decreases the value by

$$
\left[ \Phi(V[f(\cdot, t + \Delta)]) + \phi(V[f(\cdot, t + \Delta)]) \int_0^T \mu(\tau, t + \Delta) \psi(\tau, t + \Delta) d\tau' \right].
$$

Analogously, the term $\int_0^T \mu(\tau', t + \Delta) \frac{\delta \psi}{\delta f}(\tau', \tau, t + \Delta) d\tau'$ may decrease by $\Phi(V[f(\cdot, t + \Delta)])$ if the shock arrives. As $\Delta \to 0$, the Lagrange multiplier $\mu(\tau, t)$ collapses to zero given
\( \mu (\tau, t) = 0, \) for all \( \tau \in (0, T], \, t \in (0, \infty), \) \hspace{1cm} (B.12)

reflecting the lack of commitment over finite intervals. Notice that conditional on no shock arrival, the the limit case \( \Delta \to 0 \) in equation (B.8) yields to

\[ j (\tau, t) = \frac{\delta V}{\delta f} (\tau, t), \] \hspace{1cm} for all \( \tau \in (0, T], \, t \in (0, \infty), \)

which is the same equation as in the deterministic case. Taking into account (B.12) and (B.13), the limit as \( \Delta \to 0 \) of equation (B.10) can be expressed as an HJB of the form

\[ \rho j (\tau, t) = U' (c(t)) (-\delta) + \frac{\partial j}{\partial t} - \frac{\partial j}{\partial \tau} + \theta [1 - \Phi (V [f, \tau])] j (\tau, t) \]

if \( \tau \in (0, T], \)

\[ j (0, t) = -U' (c(t)) \]

if \( \tau = 0. \)

If we substitute these values in (B.10) we obtain

\[ \rho j (\tau, t) = U' (c(t)) (-\delta) + \frac{\partial j}{\partial t} - \frac{\partial j}{\partial \tau} - \theta [1 - \Phi (V [f, \tau])] j (\tau, t) \]

if \( \tau \in (0, T], \)

\[ j (0, t) = -U' (c(t)) \]

If \( \tau = 0. \)

\[ \lim_{t \to \infty} e^{-\rho t} j (\tau, t) = 0. \]

Defining again the variable

\[ v (\tau, t) = j (\tau, t) / U' (c(t)), \]

the HJB results in

\[ \left( \rho - \frac{U'' (c(t))}{U' (c(t))} \frac{dc}{dt} \right) v (\tau, t) = -\delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau} - \theta [1 - \Phi (V [f, \tau])] v (\tau, t) \]

if \( \tau \in (0, T], \)

\[ v (0, t) = -1, \] \hspace{1cm} if \( \tau = 0, \)

\[ \lim_{t \to \infty} e^{-\rho t} v (\tau, t) = 0. \]

and the first order condition is

\[ \frac{\partial q}{\partial \iota} l (\tau, t) + q (t, \tau, \iota, f) = -v (\tau, t). \]
Step 3. The aggregate HJB

The last step is to construct the aggregate HJB in order to obtain the value of \( V \left[ f (\cdot, t) \right] \). The idea is to compute the derivative with respect to \( t + \Delta \) in the dynamic programming equation \((??)\) and then to take the limit as \( \Delta \to 0 \):

\[
\rho V \left[ f (\cdot, t) \right] = U (c (t)) + \int_0^T \delta V \frac{\partial f}{\partial t} d\tau
- \theta \left\{ [1 - \Phi (V [f (\cdot, t)])] V [f (\cdot, t + \Delta)] - \Gamma (V [f (\cdot, t + \Delta)]) \right\},
\]

or equivalently

\[
\rho V \left[ f (\cdot, t) \right] = U (c (t)) + \int_0^T U' (c (t)) v (\tau, t) \frac{\partial f}{\partial t} d\tau
- \theta \left\{ [1 - \Phi (V [f (\cdot, t)])] V [f (\cdot, t + \Delta)] - \Gamma (V [f (\cdot, t + \Delta)]) \right\}.
\]

B. Proof of Corollary 5.1

The proof, follows from plugging in the valuation in the risky steady state. In particular,

\[
\iota (\tau) = \frac{\psi (\tau) + v (\tau)}{\lambda}
= \frac{1}{\lambda} \left[ -\frac{\delta \left( 1 - e^{-\{\rho + \theta [1 - \Phi (\bar{V})]\}} \right)}{\rho + \theta [1 - \Phi (\bar{V})]} - e^{-\{\rho + \theta [1 - \Phi (\bar{V})]\}} \right]
+ \frac{\delta \left( 1 - e^{-\{\rho + \theta [1 - \Phi (\bar{V})]\}} \right)}{\bar{r} + \theta [1 - \Phi (\bar{V})]} + e^{-\{\rho + \theta [1 - \Phi (\bar{V})]\}}
\]

\[
= e^{-\theta [1 - \Phi (\bar{V})]} \frac{1}{\lambda} \left[ -\frac{\delta \left( e^{\theta [1 - \Phi (\bar{V})]} - e^{-\rho \tau} \right)}{\rho + \theta [1 - \Phi (\bar{V})]} + \frac{\delta \left( e^{\theta [1 - \Phi (\bar{V})]} - e^{-\rho \tau} \right)}{\bar{r} + \theta [1 - \Phi (\bar{V})]} \right] + \left( e^{-\rho \tau} + e^{-\bar{r} \tau} \right)
\]

From Section 3, we know that the second term in square bracket are the optimal issuances when there are no shocks and the coupon rate is zero. The first term vanishes, when the coupon rate is zero.

C. Proof of Proposition 5.2

The proof is similar to that of the Proposition 1 above. The main difference is that, given the default-free value \( \bar{V} [f (\cdot, t)] \), the continuation value after the default opportunity arrives is now \( \Gamma (\bar{V} [f (\cdot, t + \Delta)]) + \Phi (\bar{V} [f (\cdot, t)]) \bar{V} [f (\cdot, t + \Delta)] \). If we proceed as in the
proof of Proposition 1 we obtain
\[
\rho j (\tau, t) = U' (c (t)) (-\delta) + \frac{\partial j}{\partial t} - \theta \left[ j (\tau, t) - \Phi (\tilde{V} [f (\cdot, t)]) \right], \text{ if } \tau \in (0, T),
\]
\[
\left. \frac{\partial}{\partial t} j (\tau, t) \right|_{\tau = 0} = -U' (c (t)), \text{ if } \tau = 0,
\]
\[
\lim_{t \to \infty} e^{-\rho t} j (\tau, t) = 0,
\]
where
\[
j (\tau, t) = \frac{\delta V}{\delta f} (\tau, t), \text{ for all } \tau \in (0, T), t \in [0, \infty),
\]
\[
\tilde{j} (\tau, t) = \frac{\delta \tilde{V}}{\delta f} (\tau, t), \text{ for all } \tau \in (0, T), t \in [0, \infty).
\]

Defining the variables
\[
v (\tau, t) = j (\tau, t) / U' (c (t)),
\]
\[
\tilde{v} (\tau, t) = \tilde{j} (\tau, t) / U' (\tilde{c} (t)),
\]
where \(\tilde{c} (t)\) is the consumption in the riskless path, the HJB before the arrival of the shock results in
\[
\left( \rho - \frac{U'' (c (t))}{U' (c (t))} \frac{dc}{dt} \right) v (\tau, t) = -\delta + \frac{\partial v}{\partial t} - \theta \left[ v (\tau, t) - \Phi (\tilde{V} [f (\cdot, t)]) \tilde{v} (\tau, t) \frac{U' (\tilde{c} (t))}{U' (c (t))} \right], \text{ if } \tau \in (0, T),
\]
\[
v (0, t) = -1, \text{ if } \tau = 0,
\]
\[
\lim_{t \to \infty} e^{-\rho t} v (\tau, t) = 0.
\]

The aggregate HJB after default is the one in the deterministic case:
\[
\rho \tilde{V} [f (\cdot, t)] = U (\tilde{c} (t)) + \int_0^T U' (\tilde{c} (t)) \tilde{v} (\tau, t) \frac{df}{dt} d\tau.
\]
C Appendix: Duality

The Primal. Given a path of resources \(y(t)\), the problem is:

\[
V[f(\cdot,0)] = \max_{\{\ell(\tau,t),c(t)\}_{\tau\in[0,\infty),t\in[0,T]}} \int_t^\infty e^{-\rho(s-t)} u(c(s))ds \text{ s.t.}
\]

\[
c(t) = y(t) - f(0,t) + \int_0^T [q(\tau,t)\ell(\tau,t) - \delta f(\tau,t)]d\tau
\]

\[
\frac{df}{dt} = \ell(\tau,t) + \frac{df}{d\tau}; f(\tau,0) = f_0(\tau)
\]

Definition C.1. Given a path of income \(\{y(t)\}\) and initial debt \(f_0\) a solution to \(P1\) is a path of consumption \(c(t)\) and debt issuances \(\ell(t,\tau)\) such that 1) the budget constraint holds for every \(t\) 2) the evolution of debt satisfies the KFE 3) the no ponzi condition holds, there is not other path of consumption and debt issuances \(\{\tilde{\ell},\tilde{\tau}\}\) that is feasible and yields strictly higher utility at zero.

Let \(j(\tau,t)\) be the lagrange multiplier associated with the KFE. It measures the change maginal utility of issuing more debt.

Proposition C.1. If a solution to \(P1\) with \(e^{-\rho t}f, e^{-\rho t}L \in L^2([0,T] \times [0,\infty)), e^{-\rho t}c \in L^2[0,\infty)\), given by \(\{\ell(\tau,t),c(t)\}_{t=0}^\infty\) exists, it satisfies the PDE

\[
\rho j(\tau,t) = \frac{\partial q}{\partial f} t - \delta + \frac{\partial j}{\partial t} - \frac{\partial j}{\partial \tau}, \text{if } \tau \in (0,T]
\]

\[
j(0,t) = -u'(c(t)), \lim_{t \to \infty} e^{-\rho t}j(\tau,t) = 0
\]

where \(v(\tau,t)\) is the marginal value of a unit of debt with time-to-maturity \(\tau\), the interest rate \(r(t)\) is given by \(r(t) = \rho - \frac{U'(|c(t)|)}{U'(c(t))}\) and \(e^{-\rho t}v \in L^2([0,T] \times [0,\infty))\); the optimal issuance \(\ell(\tau,t)\) is given by

\[
\left(\frac{\partial q}{\partial t} \ell(\tau,t) + q(t,\tau,\tau)\right)u'(c(t)) = -j(\tau,t)
\]

The Dual. This problem finds the lowest cost of achieving a particular path of consumption with the lowest amount of resources. Given a desired path of consumption \(c(t)\) the objective is to minimize the resources needed to achieve that path. More precisely, \(P2\) is given by:

\[
D[f(\cdot,0)] = \min_{\{\ell(\tau,t),y(t)\}_{\tau\in[0,\infty),t\in[0,T]}} \int_0^\infty e^{-\int_0^t r(s)ds}y(t)dt \text{ s.t.}
\]
\[ c(t) = y(t) - f(0, t) + \int_0^T [q(\tau, t, \iota(\tau, t) - \delta f(\tau, t)] d\tau \]

\[ \frac{\partial f}{\partial t} = \iota(\tau, t) + \frac{\partial f}{\partial \tau}; f(\tau, 0) = f_0(\tau) \]

\[ r(t) = \rho + \sigma \frac{\dot{c}(t)}{c(t)} \]

**Definition C.2.** Given a path of consumption \( \{c(t)\} \) and initial debt \( f_0 \) a solution to \( P_2 \) is a path of income \( y(t) \) and debt issuances \( \iota(\tau, t) \) such that 1) the budget constraint holds for every \( t \) 2) the evolution of debt satisfies the KFE 3) the no ponzi condition holds, there is not other path of consumption and debt issuances \( \{\tilde{y}, \tilde{\iota}\} \) that is feasible and has lower resources associated.

Let \( v(\tau, t) \) be the Lagrange multiplier associated with KFE. It measures marginal resources needed if a unit of debt is issued. The necessary conditions are the following:

**Proposition C.2.** If a solution to \( P_2 \) with \( e^{-\rho t} f, e^{-\rho t} \iota \in L^2([0, T] \times [0, \infty)), e^{-\rho t} c \in L^2[0, \infty) \), given by \( \{\iota(\tau, t), c(t)\} \) exists, it satisfies the PDE

\[ r(t) v(\tau, t) = \frac{\partial q}{\partial \iota} \iota(\tau, t) - \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \text{ if } \tau \in (0, T] \]

\[ v(0, t) = -1, \]

\[ \lim_{t \to \infty} e^{-\rho t} v(\tau, t) = 0 \]

where \( v(\tau, t) \) is the marginal value of a unit of debt with time-to-maturity \( \tau \), the interest rate \( r(t) \) is given by \( r(t) = \rho - \frac{U''(c(t))}{U'(c(t))} \frac{\dot{c}(t)}{c(t)} \) and \( e^{-\rho t} v \in L^2([0, T] \times [0, \infty)) \); the optimal issuance \( \iota(\tau, t) \) is given by

\[ \frac{\partial q}{\partial \iota} \iota(\tau, t) + q(t, \tau, \iota) = -v(\tau, t) \]

**Proof.** See below. 

The Connection.

**Corollary C.1.** Suppose that for a given income path \( y(t) \) and initial debt \( f_0 \) the solution to \( P_1 \) is \( c^*(t), \iota^*(\tau, t), j^*(\tau, t) \). Then, \( y(t), \iota^*(\tau, t), \frac{j^*(\tau, t)}{w(c(t))} \) solves \( P_2 \) given the path \( c^*(t) \).
C.1 Proof of Proposition

First we construct a Lagrangian in the space of functions \( g \) such that \( \|e^{-\rho t/2}g(\tau,t)\|_{L^2} < \infty \). The Lagrangian is

\[
L(t, f) = \int_0^\infty e^{-r(t)t} \left( c(t) + f(0, t) - \int_0^T [q(t, \tau, t, f) \lambda(\tau, t) - \delta f(\tau, t)] d\tau \right) dt \\
+ \int_0^\infty \int_0^T e^{-r(t)t} v(\tau, t) \left( -\frac{\partial f}{\partial t} + \lambda(\tau, t) + \frac{\partial f}{\partial \tau} \right) d\tau dt,
\]

where \( j(\tau, t) \) is the Lagrange multiplier associated to the law of motion of debt. Taking Gateaux derivatives, for any suitable \( h(\tau, t) \) such that \( e^{-\rho t} h \in L^2 ([0, T] \times [0, \infty)) \):

\[
\lim_{a \to 0} \frac{\partial}{\partial a} L(t, f + ah) = \int_0^\infty e^{-r(t)t} \left[ h(0, t) - \int_0^T \left( \frac{\partial q}{\partial f}(\tau, t) - \delta \right) h(\tau, t) d\tau \right] dt \\
+ \int_0^\infty \int_0^T e^{-r(t)t} \frac{\partial h}{\partial t} v(\tau, t) d\tau dt \\
- \int_0^\infty \int_0^T e^{-r(t)t} \frac{\partial h}{\partial \tau} v(\tau, t) d\tau dt,
\]

The last two terms can be integrated by parts

\[
- \int_0^T \frac{\partial h}{\partial \tau} v(\tau, t) d\tau = - h(T, t) v(T, t) + h(0, t) v(0, t) + \int_0^T h \frac{\partial v}{\partial \tau} d\tau,
\]

\[
+ \int_0^\infty e^{-r(t)t} \frac{\partial h}{\partial t} v(\tau, t) dt = \lim_{s \to T} e^{-r(t)t} h(\tau, s) v(\tau, s) - h(\tau, 0) v(\tau, 0) - \int_0^\infty e^{-r(t)t} h(\tau, t) \left( \frac{\partial v}{\partial t} - r(t)v \right) d\tau.
\]

As the initial distribution \( f_0 \) is given the value of \( h(\tau, 0) = 0 \). The Gateaux derivative should be zero for any suitable \( h(\tau, t) \)

\[
0 = \int_0^\infty e^{-r(t)t} \left[ +h(0, t) - \int_0^T \left( \frac{\partial q}{\partial f}(\tau, t) - \delta \right) h(\tau, t) d\tau \right] d\tau \\
- \int_0^\infty \int_0^T e^{-r(t)t} \left( -r(t)v - \frac{\partial v}{\partial \tau} + \frac{\partial v}{\partial t} \right) h(\tau, t) d\tau dt \\
- \int_0^\infty e^{-\rho t} (h(T, t) v(T, t) - h(0, t) v(0, t)) dt \\
+ \int_0^\infty \lim_{s \to \infty} e^{-\rho s} h(\tau, s) v(\tau, s) d\tau.
\]
Therefore, as \( f(T^+, t) = 0 \) then \( h(T, t) = 0 \) and we have

\[
\begin{align*}
    r(t)v(\tau, t) &= \left( \frac{\partial q}{\partial f} I - \delta \right) + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau} \quad \text{if } \tau \in (0, T) \\
    v(0, t) &= -1, \text{ if } \tau = 0, \\
    \lim_{{t \to \infty}} e^{-r(t)t}v(\tau, t) &= 0.
\end{align*}
\]

(C.1)

Proceeding similarly in the case of \( I \)

\[
\lim_{{\alpha \to 0}} \frac{\partial}{\partial \alpha} L(t + \alpha h, f) = \int_0^\infty e^{-r(t)t} \left[ - \int_0^T \left( \frac{\partial q}{\partial I} I(\tau, t) + q(t, \tau, I, f) \right) h(\tau, t) d\tau \right] dt \\
- \int_0^\infty \int_0^T e^{-r(t)t} h(\tau, t) v(\tau, t) d\tau dt,
\]

and hence

\[
\left( \frac{\partial q}{\partial I} I(\tau, t) + q(t, \tau, I, f) \right) = -v(\tau, t).
\]

The PDE equation (C.1) results in

\[
\begin{align*}
    r(t)v(\tau, t) &= \frac{\partial q}{\partial f} I - \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau} \quad \text{if } \tau \in (0, \infty), \\
    v(0, t) &= -1, \text{ if } \tau = 0, \\
    \lim_{{t \to \infty}} e^{-r(t)t}v(\tau, t) &= 0,
\end{align*}
\]

and the first order condition is

\[
\frac{\partial q}{\partial I} I(\tau, t) + q(t, \tau, I, f) = -v(\tau, t).
\]
D Computational Method Deterministic Dynamics

We describe the numerical algorithm used to jointly solve for the equilibrium value function, \( v(\tau,t) \), bond price, \( q(t,\tau,\iota) \), consumption \( c(t) \), issuance \( \iota(\tau,t) \) and density \( f(\tau,t) \). The equilibrium is characterized by the HJB equation

\[

r(t)v(\tau,t) = -\delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \quad \text{if} \quad \tau \in (0,T] \quad \text{(D.1)}
\]

\[
v(0,t) = -1, \quad \text{if} \quad \tau = 0, \quad \text{(D.2)}
\]

where the interest rate \( r(t) \) is given by:

\[
r(t) = \rho + \frac{\gamma c(t)}{c(t)}, \quad \text{(D.3)}
\]

where \( \gamma \) is the risk coefficient in \( U(c) := \frac{c^{1-\gamma} - 1}{1-\gamma} \). The optimal issuance \( \iota(\tau,t) \) is given by

\[
\iota = \frac{1}{\lambda} \left( \frac{\delta (1 - e^{-\bar{r} \tau})}{\bar{r}} + e^{-\bar{r} \tau} + v(\tau,t) \right). \quad \text{(D.4)}
\]

The law of motion of the density of maturities is given by the Kolmogorov Forward equation

\[
\frac{\partial f}{\partial t} = \iota(\tau,t) + \frac{\partial f}{\partial \tau}, \quad \text{(D.5)}
\]

and consumption by the budget constraint

\[
c(t) = \bar{y} - f(0,t) + \int_0^T \left[ \left( \frac{\delta (1 - e^{-\bar{r} \tau})}{\bar{r}} + e^{-\bar{r} \tau} - \frac{1}{2} \lambda \iota(\tau,t) \right) \iota(\tau,t) - \delta f(\tau,t) \right] d\tau. \quad \text{(D.6)}
\]

The parameters are \( T, \delta, \bar{y}, \gamma, \bar{\lambda}, \rho \) and \( \bar{r} = \rho \). The initial distribution is \( f(\tau,0) = f_0(\tau) \). The algorithm proceeds in 3 steps. We describe each step in turn.

Step 1: Solution to the Hamilton-Jacobi-Bellman equation

The HJB equation \((\text{D.1})\) is solved using an upwind finite difference scheme similar to Achdou et al. (2014). We approximate the value function \( v(\tau) \) on a finite grid with step \( \Delta \tau : \tau \in \{ \tau_1, ..., \tau_I \} \), where \( \tau_i = \tau_{i-1} + \Delta \tau = \tau_1 + (i - 1) \Delta \tau \) for \( 2 \leq i \leq I \). The bounds are \( \tau_1 = \Delta \tau \) and \( \tau_I = T \), such that \( \Delta \tau = T/I \). We use the notation \( v_i := v(\tau_i) \), and similarly for the policy function \( \iota_i \).

Notice first that the HJB equation involves first derivatives of the value function. At each point of the grid, the first derivative can be approximated with a forward or a backward approximation. In an upwind scheme, the choice of forward or backward derivative de-
pends on the sign of the drift function for the state variable. As in our case, the drift is always negative, we employ a backward approximation in state:

\[
\frac{\partial v(\tau_i)}{\partial \tau} \approx \frac{v_i - v_{i-1}}{\Delta \tau}.
\]

The HJB equation is approximated by the following upwind scheme,

\[
\rho v_i = -\delta + \frac{v_{i-1}}{\Delta \tau} - \frac{v_i}{\Delta \tau},
\]

with terminal condition \( v_0 = v(0) = -1 \). This can be written in matrix notation as

\[
\rho v = u + Av,
\]

where

\[
A = \frac{1}{\Delta \tau} \begin{bmatrix}
-1 & 0 & 0 & 0 & \cdots & 0 \\
1 & -1 & 0 & 0 & \cdots & 0 \\
0 & 1 & -1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & -1 & 0 \\
0 & 0 & \cdots & 0 & 1 & -1
\end{bmatrix}
\]

\[
v = \begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
\vdots \\
v_{I-1} \\
v_I
\end{bmatrix}
\]

\[
\]

The solution is given by

\[
v = (\rho I - A)^{-1} u.
\]

Most computer software packages, such as Matlab, include efficient routines to handle sparse matrices such as \( A \).

To analyze the transitional dynamics, define \( t_{\text{max}} \) as the time interval considered, which should be large enough to ensure a converge to the stationary distribution and time is discretized as \( t_n = t_{n-1} + \Delta t \), in intervals of length

\[
\Delta t = \frac{t_{\text{max}}}{N-1}.
\]
where $N$ is a constant. We use now the notation $v^n_i := v(\tau_i, t_n)$. The value function at $t^\text{max}$ is the stationary solution computed in (D.9) that we denote as $v^N$.\footnote{You may begin directly by employing the analytical solution from equation (2.22) as $v^N$.} We choose a forward approximation in time. The dynamic HJB (D.1) can thus be expressed

\[
 r^n v^n = u + A v^n + \frac{(v^{n+1} - v^n)}{\Delta t},
\]

where $r^n := r(t_n)$. By defining $B^n = \left(\frac{1}{\Delta t} + r^n\right) I - A$ and $d^{n+1} = u + \frac{v^{n+1}}{\Delta t}$, we have

\[
v^n = (B^n)^{-1} d^{n+1}, \tag{D.10}
\]

which can be solved backwards from $n = N - 1$ until $n = 1$.

The optimal issuance is given by

\[
\iota^n_i = \frac{1}{\lambda} (\psi_i + v^n_i),
\]

where

\[
\psi_i = \frac{\delta (1 - e^{-\rho \tau_i})}{\rho} + e^{-\rho \tau_i}.
\]

**Step 2: Solution to the Kolmogorov Forward equation**  Analogously, the KFE equation (2.1) can be approximated as

\[
\frac{f^n_i - f^{n-1}_i}{\Delta t} = \iota^n_i + \frac{f^n_{i+1} - f^n_i}{\Delta \tau},
\]

where we have employed the notation $f^n_i := f(\tau_i, t_n)$. This can be written in matrix notation as:

\[
\frac{f^n - f^{n-1}}{\Delta t} = -\tau + A^T f^n, \tag{D.11}
\]

where $A^T$ is the transpose of $A$ and

\[
f^n = \begin{bmatrix} f^n_1 \\ f^n_2 \\ \vdots \\ f^n_{i-1} \\ f^n_i \end{bmatrix}.
\]
Given \( f_0 \), the discretized approximation to the initial distribution \( f_0(\tau) \), we can solve the KF equation forward as

\[
f_n = \left( I - \Delta t A^T \right)^{-1} \left( ^n \Delta t + f_{n-1} \right), \quad n = 1, \ldots, N. \tag{D.12}
\]

**Step 3: Computation of consumption** The discretized budget constraint (??) can be expressed as

\[
c^n = \bar{y} - f_1^{n-1} \Delta \tau + \sum_{i=1}^{I} \left[ \left( \psi_i - \frac{1}{2} \lambda_i^n \right) i_i^n - \delta f_i^n \right] \Delta \tau, \quad n = 1, \ldots, N.
\]

Compute

\[
r^n = \rho + \gamma \frac{c^{n+1} - c^n}{\Delta t}, n = 1, \ldots, N - 1.
\]

**Complete algorithm** The algorithm proceeds as follows. First guess an initial path for consumption, for example \( c^n = \bar{y} \), for \( n = 1, \ldots, N \). Set \( k = 1 \);

**Step 1: HJB.** Given \( c_{k-1} \) solve the HJB and obtain \( \iota \).

**Step 2: KF.** Given \( \iota \) solve the KF equation with initial distribution \( f_0 \) and obtain the distribution \( f \).

**Step 3: Consumption.** Given \( \iota \) and \( f \) compute consumption \( c \). If \( \|c - c_{k-1}\| = \sum_{n=1}^{N} |c^n - c_{k-1}^n| < \varepsilon \) then stop. Otherwise compute

\[
c_k = \omega c + (1 - \omega) c_{k-1}, \quad \lambda \in (0, 1),
\]

set \( k := k + 1 \) and return to step 1.